On polyvectors of vector spaces and hyperplanes of projective Grassmannians

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Abstract

We investigate relationships between polyvectors of a vector space V, alternating multilinear forms on V, hyperplanes of projective Grassmannians and regular spreads of projective spaces. Suppose V is an n-dimensional vector space over a field $\mathbb{F}$ and that $A_{n-1,k}(\mathbb{F})$ is the Grassmannian of the $(k-1)$-dimensional subspaces of $\mathrm{PG}(V)$ $(1 \leq k \leq n-1)$. With each hyperplane H of $A_{n-1,k}(\mathbb{F})$, we associate an $(n-k)$-vector of V (i.e., a vector of $\bigwedge^{n-k} V$) which we will call a representative vector of H. One of the problems which we consider is the isomorphism problem of hyperplanes of $A_{n-1,k}(\mathbb{F})$, i.e. how isomorphism of hyperplanes can be recognized in terms of their representative vectors. Special attention is paid here to the case $n = 2k$ and to those isomorphisms which arise from dualities of $\mathrm{PG}(V)$. We also prove that with each regular spread of the projective space $\mathrm{PG}(2k-1, \mathbb{F})$, there is associated some class of isomorphic hyperplanes of the Grassmannian $A_{2k-1,k}(\mathbb{F})$, and we study some properties of these hyperplanes. The above investigations allow us to obtain a new proof for the classification, up to equivalence, of the trivectors of a 6-dimensional vector space over an arbitrary field $\mathbb{F}$, and to obtain a classification, up to isomorphism, of all hyperplanes of $A_{5,3}(\mathbb{F})$.

Keywords: polyvector, multilinear form, projective Grassmannian, hyperplane, regular spread
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1 Overview

The aim of this paper is to investigate relationships between polyvectors of an n-dimensional vector space V over a field $\mathbb{F}$, alternating multilinear forms on V, hyperplanes of projective Grassmannians defined on $\mathrm{PG}(V)$, and regular spreads of $\mathrm{PG}(V)$.

Suppose $\dim(V) = n \geq 2$ and $k \in \{1, \ldots, n-1\}$. With every hyperplane H of the Grassmannian $A_{n-1,k}(\mathbb{F})$ of the $(k-1)$-dimensional subspaces of the projective space $\mathrm{PG}(V)$, we will associate an $(n-k)$-vector of V, which we call a *representative vector* of H. This $(n-k)$-vector is determined up to a nonzero factor of $\mathbb{F}$.

One of the problems which we will address is the isomorphism problem of hyperplanes of $A_{n-1,k}(\mathbb{F})$. Suppose H_1 and H_2 are two hyperplanes of $A_{n-1,k}(\mathbb{F})$ and that $\alpha_1 \in \bigwedge^{n-k} V$ and $\alpha_2 \in \bigwedge^{n-k} V$ are representative vectors of H_1 and H_2, respectively. Then there exists an automorphism of $A_{n-1,k}(\mathbb{F})$ induced by a projectivity of $\mathrm{PG}(V)$ mapping H_1 to H_2 if and only if the vector α_1 is equivalent with a nonzero multiple of α_2 (which means that there is an element of $GL(\bigwedge^{n-k} V)$ induced by an element of $GL(V)$ which maps α_1 to a nonzero multiple of α_2). However, in many cases there are much more automorphisms than just those which arise from projectivities. There are also automorphisms which are associated to collineations of $\mathrm{PG}(V)$ whose corresponding field automorphisms are non-trivial, and in the case $n = 2k$, there are also automorphisms which arise from dualities of $\mathrm{PG}(V)$. We are especially interested in the latter case. Since the group of automorphisms of $A_{2k-1,k}(\mathbb{F})$ which are induced by collineations of $\mathrm{PG}(V)$ is a (normal) subgroup of index 2 of the full automorphism group of $A_{2k-1,k}(\mathbb{F})$, it suffices to take one particular isomorphism η of $A_{2k-1,k}(\mathbb{F})$ which is associated to some duality of $\mathrm{PG}(V)$, and consider the following problem:

> Suppose $\alpha \in \bigwedge^{k} V$ is a representative vector of the hyperplane H of $A_{2k-1,k}(\mathbb{F})$. Derive from α a representative vector of the hyperplane H^{η} of $A_{2k-1,k}(\mathbb{F})$.

The investigation of this problem led us to the notion of *dual vector of α with respect to some ordered basis B of V*. We will investigate this notion in Section 3. The isomorphism problem for the hyperplanes of $A_{n-1,k}(\mathbb{F})$ itself will be investigated in Section 5.

Suppose $n = 2k$ and that S is a regular spread of $\mathrm{PG}(V)$. Let X denote the set of all $(k-1)$-dimensional subspaces of $\mathrm{PG}(V)$ which contain at least 1 line of S, and let $\mathcal{H}$ denote the set of all hyperplanes of $A_{n-1,k}(\mathbb{F})$ containing X. Then we will show in Section 6 that every two distinct hyperplanes of $\mathcal{H}$ are isomorphic. Moreover, the representative vectors which correspond to the elements of $\mathcal{H}$ are precisely the nonzero vectors of a certain two-dimensional subspace of $\bigwedge^{k} V$. Some other properties of these hyperplanes will be examined.

The above results will allow us in Section 7.4 to obtain an alternative proof for the classification, up to equivalence, of the trivectors of a 6 dimensional vector space over an arbitrary field $\mathbb{F}$. This classification is originally due to Revoy [15] and a number of other authors have obtained classifications for some special classes of fields, see [4, 6, 10, 11, 14]. The methods which we will use in Section 7.4 were suggested to the author while examining some geometrical properties of the associated hyperplanes of $A_{5,3}(\mathbb{F})$ (see e.g. Proposition 7.10). The classification, up to isomorphism, of the hyperplanes of $A_{5,3}(\mathbb{F})$ can be found in Proposition 7.9.

2 The connection between polyvectors, alternating multilinear forms and hyperplanes of Grassmannians

2.1 Polyvectors

Let $n \in \mathbb{N} \setminus \{0\}$ and $k \in \{0, \ldots, n\}$. Let V be an n-dimensional vector space over a field $\mathbb{F}$ and let $\bigwedge^k V$ denote the k-th exterior power of V ($\bigwedge^0 V = \mathbb{F}$; $\bigwedge^1 V = V$). The elements of $\bigwedge^k V$ are also called the *k-vectors* of V. A *polyvector* of V is a k'-vector for some $k' \in \{0, \ldots, n\}$.

Suppose $k \in \{1, \ldots, n\}$. Then for every $\theta \in GL(V)$, there exists a unique $\bigwedge^k(\theta) \in GL(\bigwedge^k V)$ such that $\bigwedge^k(\theta)(\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_k) = \theta(\bar{v}_1) \wedge \theta(\bar{v}_2) \wedge \cdots \wedge \theta(\bar{v}_k)$ for all vectors $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k$ of V. Two k-vectors α_1 and α_2 of V are called *equivalent* if there is a $\theta \in GL(V)$ such that $\bigwedge^k(\theta)(\alpha_1) = \alpha_2$. The k-vectors α_1 and α_2 are called *semi-equivalent* if α_1 is equivalent with some nonzero multiple of α_2. Regarding the classification of polyvectors, the following results can be found in the literature.

- Suppose $n \geq 2$. Up to equivalence, there is one nonzero 1-vector, one nonzero $(n-1)$-vector and one nonzero n-vector of V.

- Suppose $n \geq 2$. There are $\lfloor \frac{n}{2} \rfloor$ equivalence classes of nonzero bivectors of V. If $\{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n\}$ is a basis of V, then the bivectors $\sum_{i=1}^{k} \bar{e}_{2i-1} \wedge \bar{e}_{2i}$, $k \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$, are representatives of these $\lfloor \frac{n}{2} \rfloor$ classes.

- Suppose V is an n-dimensional complex vector space. A classification of the trivectors of V was obtained in Reichel [14] for the case $n = 6$, in Schouten [17] for the case $n = 7$, in Gurevich [12] for the case $n = 8$ and in Vinberg & Èlašvili [19] for the case $n = 9$. A summary of the results obtained for the cases $n \in \{6, 7, 8\}$ can be found in Gurevich [13, §35].

- Suppose V is an n-dimensional real vector space. A classification of the trivectors of V was obtained in Gurevich [10, 11] and Capdevielle [4] for the case $n = 6$, in Westwick [20] for the case $n = 7$ and in Djoković [9] for the case $n = 8$.

- Suppose V is a vector space of dimension $n \in \{6, 7\}$ over a perfect field of cohomological dimension at most 1. A classification of the trivectors of V was obtained in Cohen & Helminck [6].

- Suppose V is a vector space over an arbitrary field $\mathbb{F}$. A classification of the trivectors of V was obtained in Revoy [15] for the case $n = 6$ and in Revoy [16] for the case $n = 7$.

2.2 Alternating multilinear forms

Let V be a vector space of dimension $n \geq 0$ over a field $\mathbb{F}$ and let $k \in \mathbb{N}\backslash\{0\}$. A *alternating k-linear form* on V is a map $f : V^k \to \mathbb{F}$ which satisfies the following properties:

(1) f is linear in each of its components;

(2) $f(\bar{v}_{\sigma(1)}, \bar{v}_{\sigma(2)}, \ldots, \bar{v}_{\sigma(k)}) = sgn(\sigma) \cdot f(\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k)$ for all vectors $\bar{v}_1, \ldots, \bar{v}_k$ of V and every permutation σ of $\{1, \ldots, k\}$.

Notice that if $k > n$, then every alternating k-linear map on V is the zero map. In the sequel, we will suppose that $n \geq 2$ and $k \in \{1, \ldots, n-1\}$.

Let ξ be a given nonzero vector of $\bigwedge^n V$ and let α be a given vector of $\bigwedge^{n-k} V$. Then for all $\bar{v}_1, \ldots, \bar{v}_k \in V$, we define $f_{\alpha,\xi}(\bar{v}_1, \ldots, \bar{v}_k)$ by:

$$\alpha \wedge \bar{v}_1 \wedge \cdots \wedge \bar{v}_k \quad = \quad f_{\alpha,\xi}(\bar{v}_1, \ldots, \bar{v}_k) \cdot \xi.$$

Then, clearly $f_{\alpha,\xi}$ is an alternating k-linear form on V. We have $f_{\lambda_1 \cdot \alpha_1 + \lambda_2 \cdot \alpha_2, \xi} = \lambda_1 \cdot f_{\alpha_1,\xi} + \lambda_2 \cdot f_{\alpha_2,\xi}$ for all $\lambda_1, \lambda_2 \in \mathbb{F}$, for all $\alpha_1, \alpha_2 \in \bigwedge^{n-k} V$ and every nonzero $\xi \in \bigwedge^n V$. Also, $f_{\alpha,\lambda\cdot\xi} = \frac{1}{\lambda} f_{\alpha,\xi}$ for every $\lambda \in \mathbb{F} \setminus \{0\}$, for every $\alpha \in \bigwedge^{n-k} V$ and every nonzero $\xi \in \bigwedge^n V$.

For every alternating k-linear form f on V and for every nonzero $\xi \in \bigwedge^n V$, there is a unique $\alpha \in \bigwedge^{n-k} V$ such that $f = f_{\alpha,\xi}$. To see this, take a basis $\{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n\}$ of V and let $\lambda \in \mathbb{F} \setminus \{0\}$ such that $\xi = \lambda \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \cdots \wedge \bar{e}_n$. For all $i_1, \ldots, i_k, j_1, \ldots, j_{n-k}$ satisfying $\{i_1, \ldots, i_k, j_1, \ldots, j_{n-k}\} = \{1, 2, \ldots, n\}$, $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_{n-k}$, we define

$$a(j_1, \ldots, j_{n-k}) \quad := \quad sgn\begin{pmatrix} 1 & \cdots & n-k & n-k+1 & \cdots & n \\ j_1 & \cdots & j_{n-k} & i_1 & \cdots & i_k \end{pmatrix} \cdot \lambda \cdot f(\bar{e}_{i_1}, \ldots, \bar{e}_{i_k}), \tag{1}$$

and

$$\alpha \quad := \quad \sum a(j_1, \ldots, j_{n-k}) \cdot \bar{e}_{j_1} \wedge \cdots \wedge \bar{e}_{j_{n-k}}, \tag{2}$$

where the summation ranges over all $j_1, j_2, \ldots, j_{n-k} \in \{1, \ldots, n\}$ satisfying $j_1 < j_2 < \cdots < j_{n-k}$. Then we necessarily have $f = f_{\alpha,\xi}$ since $f(\bar{e}_{i_1}, \ldots, \bar{e}_{i_k}) = f_{\alpha,\xi}(\bar{e}_{i_1}, \ldots, \bar{e}_{i_k})$ for all $i_1, i_2, \ldots, i_k \in \{1, \ldots, n\}$ satisfying $i_1 < i_2 < \cdots < i_k$. The uniqueness of α is also readily verified. The fact that $f(\bar{e}_{i_1}, \ldots, \bar{e}_{i_k}) = f_{\alpha,\xi}(\bar{e}_{i_1}, \ldots, \bar{e}_{i_k})$ for all $i_1, i_2, \ldots, i_k \in \{1, \ldots, n\}$ satisfying $i_1 < i_2 < \cdots < i_k$, implies that α must be as defined in equation (2).

Let V^* denote the dual space of V. Then by Bourbaki [2, §8.2] $\bigwedge^k V^*$ can be regarded as the dual space of $\bigwedge^k V$ by putting $(\omega_1 \wedge \cdots \wedge \omega_k)(\bar{v}_1 \wedge \cdots \wedge \bar{v}_k)$ equal to $\det(\omega_i(\bar{v}_j)) = \sum_\sigma sgn(\sigma) \prod_{i=1}^k \omega_i(\bar{v}_{\sigma(i)})$ for all $\omega_1, \omega_2, \ldots, \omega_k \in V^*$ and all $\bar{v}_1, \ldots, \bar{v}_k \in V$ (and extending linearly). The summation in the above sum ranges over all permutations σ of $\{1, \ldots, k\}$. Now, for every $\theta \in GL(V)$, we define a $\theta^* \in GL(V^*)$ by putting $\theta^*(\omega)(\bar{v}) = \omega(\theta^{-1}(\bar{v}))$ for all $\omega \in V^*$ and all $\bar{v} \in V$. Clearly, we have $(\theta_1 \circ \theta_2)^* = \theta_1^* \circ \theta_2^*$ and $(\theta^*)^{-1} = (\theta^{-1})^*$ for all $\theta, \theta_1, \theta_2 \in GL(V)$. Also, if I denotes the identity element of $GL(V)$, then I^* is the identity element of $GL(V^*)$. For every $\theta \in GL(V)$, for every $\alpha \in \bigwedge^k V$ and every $\chi \in \bigwedge^k V^*$, we have $\chi(\alpha) = \bigwedge^k(\theta^*)(\chi)\Big(\bigwedge^k(\theta)(\alpha)\Big)$.

If $\chi \in \bigwedge^k V^*$, then we define

$$f_\chi(\bar{v}_1, \ldots, \bar{v}_k) \quad = \quad \chi(\bar{v}_1 \wedge \cdots \wedge \bar{v}_k)$$

for all vectors $\bar{v}_1, \ldots, \bar{v}_k \in V$. Then f_χ is an alternating k-linear form of V. Clearly, $f_{\lambda_1 \cdot \chi_1 + \lambda_2 \cdot \chi_2} = \lambda_1 \cdot f_{\chi_1} + \lambda_2 \cdot f_{\chi_2}$ for all $\lambda_1, \lambda_2 \in \mathbb{F}$ and all $\chi_1, \chi_2 \in \bigwedge^k V^*$.

Conversely, if f is an alternating k-linear form on V, then there is a unique $\chi \in \bigwedge^k V^*$ such that $f = f_\chi$: if $(\bar{e}_1, \ldots, \bar{e}_n)$ is an ordered basis of V and if $(\omega_1, \ldots, \omega_n)$ denotes the corresponding dual basis of V^*, then necessarily

$$\chi \quad = \quad \sum f(\bar{e}_{i_1}, \bar{e}_{i_2}, \ldots, \bar{e}_{i_k}) \cdot \omega_{i_1} \wedge \omega_{i_2} \wedge \cdots \wedge \omega_{i_k}, \tag{3}$$

where the summation ranges over all $i_1, i_2, \ldots, i_k \in \{1, \ldots, n\}$ satisfying $i_1 < i_2 < \cdots < i_k$.

Two alternating k-linear forms $f_1 : V^k \to \mathbb{K}$ and $f_2 : V^k \to \mathbb{K}$ are called *equivalent* if there exists a $\theta \in GL(V)$ such that $f_2(\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k) = f_1(\theta(\bar{v}_1), \theta(\bar{v}_2), \ldots, \theta(\bar{v}_k))$ for all $\bar{v}_1, \ldots, \bar{v}_k \in V$. The alternating k-linear forms $f_1 : V^k \to \mathbb{K}$ and $f_2 : V^k \to \mathbb{K}$ are called *semi-equivalent* if there exists a $\theta \in GL(V)$ and a $\lambda \in \mathbb{F} \setminus \{0\}$ such that f_2 is equivalent with $\lambda \cdot f_1$.

Proposition 2.1 (1) *Let $\chi_1, \chi_2 \in \bigwedge^k V^*$. Then f_{χ_1} and f_{χ_2} are equivalent if and only if χ_1, χ_2 are equivalent.*

(2) *Let $\xi \in \bigwedge^n V \setminus \{0\}$ and $\alpha_1, \alpha_2 \in \bigwedge^{n-k} V$. If $\theta \in GL(V)$ such that $\alpha_2 = \bigwedge^{n-k}(\theta)(\alpha_1)$, then $\det(\theta) \cdot f_{\alpha_1,\xi}$ and $f_{\alpha_2,\xi}$ are equivalent.*

(3) *Let $\xi \in \bigwedge^n V \setminus \{0\}$ and $\alpha_1, \alpha_2 \in \bigwedge^{n-k} V$. Then α_1 and α_2 are semi-equivalent if and only if $f_{\alpha_1,\xi}$ and $f_{\alpha_2,\xi}$ are semi-equivalent.*

Proof. (1) For every $\theta \in GL(V)$, for every $\chi \in \bigwedge^k V^*$ and all $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k \in V$, we have $f_\chi(\bar{v}_1, \ldots, \bar{v}_k) = \chi(\bar{v}_1 \wedge \ldots \wedge \bar{v}_k) = \bigwedge^k(\theta^*)(\chi)(\theta(\bar{v}_1) \wedge \cdots \wedge \theta(\bar{v}_k)) = f_{\bigwedge^k(\theta^*)(\chi)}(\theta(\bar{v}_1), \ldots, \theta(\bar{v}_k))$. So, f_χ and $f_{\bigwedge^k(\theta^*)(\chi)}$ are equivalent. This proves the "if" part of Claim (1). Conversely, suppose $\chi_1, \chi_2 \in \bigwedge^k V^*$ such that f_{χ_1} and f_{χ_2} are equivalent. Then there exists a $\theta \in GL(V)$ such that $f_{\chi_2}(\theta(\bar{v}_1), \ldots, \theta(\bar{v}_k)) = f_{\chi_1}(\bar{v}_1, \ldots, \bar{v}_k) = f_{\bigwedge^k(\theta^*)(\chi_1)}(\theta(\bar{v}_1), \ldots, \theta(\bar{v}_2))$ for all $\bar{v}_1, \ldots, \bar{v}_k \in V$. This implies that $f_{\chi_2} = f_{\bigwedge^k(\theta^*)(\chi_1)}$. Hence, $\chi_2 = \bigwedge^k(\theta^*)(\chi_1)$ and χ_1 is equivalent with χ_2.

(2) We have $f_{\alpha_2,\xi}(\theta(\bar{v}_1), \ldots, \theta(\bar{v}_k)) \cdot \xi = \bigwedge^{n-k}(\theta)(\alpha_1) \wedge \theta(\bar{v}_1) \wedge \cdots \wedge \theta(\bar{v}_k) = \det(\theta) \cdot \alpha_1 \wedge \bar{v}_1 \wedge \cdots \wedge \bar{v}_k = \det(\theta) \cdot f_{\alpha_1,\xi}(\bar{v}_1, \ldots, \bar{v}_k) \cdot \xi$ for all vectors $\bar{v}_1, \ldots, \bar{v}_k$ of V. So, $\det(\theta) \cdot f_{\alpha_1,\xi}$ and $f_{\alpha_2,\xi}$ are equivalent.

(3) If α_1 and α_2 are semi-equivalent, then also $f_{\alpha_1,\xi}$ and $f_{\alpha_2,\xi}$ are semi-equivalent by (2). Conversely, suppose that $f_{\alpha_1,\xi}$ and $f_{\alpha_2,\xi}$ are semi-equivalent. Let $\theta \in GL(V)$ and $\lambda \in \mathbb{F} \setminus \{0\}$ such that $f_{\alpha_2,\xi}(\theta(\bar{v}_1), \ldots, \theta(\bar{v}_k)) = \lambda \cdot f_{\alpha_1,\xi}(\bar{v}_1, \ldots, \bar{v}_k)$ for all $\bar{v}_1, \ldots, \bar{v}_k \in V$. By the discussion in (2), $f_{\alpha_1,\xi}(\bar{v}_1, \ldots, \bar{v}_k) = \frac{1}{\det(\theta)} \cdot f_{\alpha_1',\xi}(\theta(\bar{v}_1), \ldots, \theta(\bar{v}_k))$, where $\alpha_1' = \bigwedge^{n-k}(\theta)(\alpha_1)$. It follows that $f_{\alpha_2,\xi}(\theta(\bar{v}_1), \ldots, \theta(\bar{v}_k)) = \frac{\lambda}{\det(\theta)} \cdot f_{\alpha_1',\xi}(\theta(\bar{v}_1), \ldots, \theta(\bar{v}_k))$ for all

$\bar{v}_1, \ldots, \bar{v}_k \in V$. So, $f_{\alpha_2,\xi} = \frac{\lambda}{\det(\theta)} \cdot f_{\alpha_1',\xi}$. It follows that $\alpha_2 = \frac{\lambda}{\det(\theta)}\alpha_1'$. Hence, α_1 and α_2 are semi-equivalent. ■

By Proposition 2.1(1), the problem of determining the (semi-)equivalence classes of k-vectors of V (or equivalently, of V^*) is equivalent to the problem of determining the (semi-)equivalence classes of alternating k-linear forms on V. By Proposition 2.1(3), the problem of determining the semi-equivalence classes of $(n-k)$-vectors of V is equivalent to the problem of determining the semi-equivalence classes of alternating k-linear forms on V and hence equivalent with the problem of determining the semi-equivalence classes of k-vectors of V. A similar conclusion does not necessarily hold for the equivalence classes, see e.g. (the final example of) Section 4.

2.3 Hyperplanes of projective Grassmannians

Let $\mathbb{F}$ be a field, $n \in \mathbb{N} \setminus \{0, 1\}$ and $k \in \{1, \ldots, n-1\}$. Let V be an n-dimensional vector space over $\mathbb{F}$ and let $PG(V) \cong PG(n-1, \mathbb{F})$ denote the corresponding projective space. We define the following point-line geometry $A_{n-1,k}(\mathbb{F})$:

- The points of $A_{n-1,k}(\mathbb{F})$ are the $(k-1)$-dimensional subspaces of $PG(V)$.
- The lines of $A_{n-1,k}(\mathbb{F})$ are the sets $L(\pi_1, \pi_2)$ of $(k-1)$-dimensional subspaces of $PG(V)$ which contain a given $(k-2)$-dimensional subspace π_1 and are contained in a given k-dimensional subspace π_2 ($\pi_1 \subset \pi_2$).
- Incidence is containment.

The geometry $A_{n-1,k}(\mathbb{F})$ is called the *Grassmannian* of the $(k-1)$-dimensional subspaces of $PG(V)$. Obviously, $A_{n-1,k}(\mathbb{F}) \cong A_{n-1,n-k}(\mathbb{F})$ and the geometry $A_{n-1,1}(\mathbb{F}) \cong A_{n-1,n-1}(\mathbb{F})$ is isomorphic to (the point-line system of) the projective space $PG(n-1, \mathbb{F})$. A *hyperplane* of $A_{n-1,k}(\mathbb{F})$ is a proper set of points of $A_{n-1,k}(\mathbb{F})$ meeting each line in either a singleton or the whole line. For a proof of the following proposition, see e.g. De Bruyn [8, Lemma 2.1].

Proposition 2.2 *Every hyperplane of $A_{n-1,k}(\mathbb{F})$ is a maximal (proper) subspace.*

For every point $p = \langle \bar{v}_1, \ldots, \bar{v}_k \rangle$ of $A_{n-1,k}(\mathbb{F})$, let $e_{gr}(p)$ denote the point $\langle \bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_k \rangle$ of $PG(\bigwedge^k V)$. The map e_{gr} defines a projective embedding of the geometry $A_{n-1,k}(\mathbb{F})$ into the projective space $PG(\bigwedge^k V)$, which is called the *Grassmann embedding* of $A_{n-1,k}(\mathbb{F})$. If π is a hyperplane of $PG(\bigwedge^k V)$, then the set H_π of all points p of $A_{n-1,k}(\mathbb{F})$ for which $e_{gr}(p) \in \pi$ is clearly a hyperplane of $A_{n-1,k}(\mathbb{F})$.

The following proposition is known, see e.g. Shult [18].

Proposition 2.3 (1) *Let f be a nonzero alternating k-linear form on V. Then the set H_f of all $(k-1)$-dimensional subspaces $\langle \bar{v}_1, \bar{v}_2, \dots, \bar{v}_k \rangle$ of* $\mathrm{PG}(V)$ *for which $f(\bar{v}_1, \bar{v}_2, \dots, \bar{v}_k) = 0$ is a hyperplane of $A_{n-1,k}(\mathbb{F})$.*

(2) *If f_1 and f_2 are two nonzero alternating k-linear forms on V, then $H_{f_1} = H_{f_2}$ if and only if f_2 is a nonzero multiple of f_1.*

Proof. (1) Observe first that if $\{\bar{v}_1, \dots, \bar{v}_k\}$ and $\{\bar{v}'_1, \dots, \bar{v}'_k\}$ generate the same $(k-1)$-dimensional projective space of $\mathrm{PG}(V)$, then $f(\bar{v}_1, \bar{v}_2, \dots, \bar{v}_k) = 0$ if and only if $f(\bar{v}'_1, \bar{v}'_2, \dots, \bar{v}'_k) = 0$. So, the set H_f is well-defined. Notice also that since f is nonzero, H_f is a proper set of points of $A_{n-1,k}(\mathbb{F})$.

Consider a line $L(\pi_1, \pi_2)$ of $A_{n-1,k}(\mathbb{F})$. We can choose vectors $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_k, \bar{v}_{k+1}$ in V such that $\pi_1 = \langle \bar{v}_1, \dots, \bar{v}_{k-1} \rangle$ and $\pi_2 = \langle \bar{v}_1, \dots, \bar{v}_{k+1} \rangle$. Then $L(\pi_1, \pi_2) = \{\langle \bar{v}_1, \dots, \bar{v}_k \rangle\} \cup \{\langle \bar{v}_1, \dots, \bar{v}_{k+1} + \lambda \bar{v}_k \rangle \mid \lambda \in \mathbb{F}\}$. Since $f(\bar{v}_1, \dots, \bar{v}_{k+1} + \lambda \bar{v}_k) = f(\bar{v}_1, \dots, \bar{v}_{k+1}) + \lambda \cdot f(\bar{v}_1, \dots, \bar{v}_k)$, it is easily seen that either one or all points of $L(\pi_1, \pi_2)$ are contained in H_f. So, H_f is a hyperplane of $A_{n-1,k}(\mathbb{F})$.

(2) Clearly, $H_{f_1} = H_{f_2}$ if f_2 is a nonzero multiple of f_1. Conversely, suppose that $H = H_{f_1} = H_{f_2}$ and let $p = \langle \bar{v}_1, \dots, \bar{v}_k \rangle$ be a point of $A_{n-1,k}(\mathbb{F})$ not contained in H. Then $f_1(\bar{v}_1, \dots, \bar{v}_k) \neq 0 \neq f_2(\bar{v}_1, \dots, \bar{v}_k)$. So, there exists a $\lambda \neq 0$ such that $(f_2 - \lambda \cdot f_1)(\bar{v}_1, \dots, \bar{v}_k) = 0$. If $f_2 \neq \lambda \cdot f_1$, then $H \cup \{p\} \subseteq H_{f_2 - \lambda \cdot f_1}$, in contradiction with the fact that H is a maximal subspace of $A_{n-1,k}(\mathbb{F})$ (recall Proposition 2.2). So, $f_2 = \lambda \cdot f_1$ is a nonzero multiple of f_1. ■

The two (equivalent) statements in the following proposition are the main results of Shult [18] (see also De Bruyn [8] for a shorter proof).

Proposition 2.4 (1) *For every hyperplane H of $A_{n-1,k}(\mathbb{F})$, there exists a nonzero alternating k-linear form f such that $H = H_f$.*

(2) *If H is a hyperplane of $A_{n-1,k}(\mathbb{F})$, then $H = H_\pi$ for a unique hyperplane π of $\mathrm{PG}(\bigwedge^k V)$.*

Definition. If H is a hyperplane of $A_{n-1,k}(\mathbb{F})$, then there exists a nonzero alternating k-linear form f on V such that $H = H_f$, and nonzero vectors $\alpha \in \bigwedge^{n-k} V$ and $\xi \in \bigwedge^n V$ such that $f = f_{\alpha,\xi}$. Notice here that f and α are uniquely determined up to nonzero factors. We call (any nonzero factor of) α a *representative vector* of the hyperplane H.

Remarks. (1) Propositions 2.3 and 2.4(1) say that there is a one-to-one correspondence between the set of hyperplanes of $A_{n-1,k}(\mathbb{F})$ and the scalar classes of nonzero alternating k-linear forms on V (two nonzero alternating k-linear forms are said to belong to the same *scalar class* if each of them is a nonzero multiple of the other). In the special case $k = 2$, this result was also obtained in Cooperstein and Shult [7].

(2) Suppose π is a hyperplane of $\mathrm{PG}(\bigwedge^k V)$. It is easily seen that there exists a nonzero vector $\alpha \in \bigwedge^{n-k} V$ such that a point $\langle \beta \rangle$ of $\mathrm{PG}(\bigwedge^k V)$ belongs to π if and only if $\alpha \wedge \beta = 0$ (make the calculations with respect to some fixed ordered basis of V). If ξ is some nonzero vector of $\bigwedge^n V$, then we obviously have $H_\pi = H_{f_{\alpha,\xi}}$. The correspondence $\pi \leftrightarrow f_{\alpha,\xi}$ defines

a bijective correspondence between the set of hyperplanes of $\mathrm{PG}(\bigwedge^k V)$ and the scalar classes of nonzero alternating k-linear forms on V. This explains why the two statements in Proposition 2.4 can be regarded as equivalent.

3 Dual vectors with respect to some ordered basis

We continue with the notations introduced in Section 2.2. Recall that V is a vector space of dimension $n \geq 2$ over a field $\mathbb{F}$ and that $k \in \{1, \dots, n-1\}$.

Definitions. (1) Let $B = (\bar{e}_1, \dots, \bar{e}_n)$ be an ordered basis of V and let $B^* = (\omega_1, \dots, \omega_n)$ denote the corresponding dual basis of V^*. Then ρ_B denotes the linear isomorphism between V and V^* defined by $\bar{e}_i \mapsto \omega_i$, $i \in \{1, \dots, n\}$. The linear isomorphism $\rho_B : V \to V^*$ induces a unique linear isomorphism $\rho_{B,k}$ between $\bigwedge^k V$ and $\bigwedge^k V^*$ which satisfies $\rho_{B,k}(\bar{e}_{i_1} \wedge \bar{e}_{i_2} \wedge \cdots \wedge \bar{e}_{i_k}) = \omega_{i_1} \wedge \omega_{i_2} \wedge \cdots \wedge \omega_{i_k}$ for all $i_1, i_2, \dots, i_k \in \{1, \dots, n\}$.

(2) Using the connection between the vectors of $\bigwedge^k V^*$, the alternating k-linear forms and the vectors of $\bigwedge^{n-k} V$, we readily see that there is a natural bijective correspondence Φ_k^* between the 1-dimensional subspaces of $\bigwedge^k V^*$ and the 1-dimensional subspaces of $\bigwedge^{n-k} V$. Let U be a 1-dimensional subspace of $\bigwedge^k V^*$ and let $\chi \in \bigwedge^k V^*$ such that $U = \langle \chi \rangle$. If we fix $\xi \in \bigwedge^n V \setminus \{0\}$, then there is a unique $\alpha \in \bigwedge^{n-k} V$ such that $f_{\alpha,\xi} = f_\chi$. We define $\Phi_k^*(U) := \langle \alpha \rangle$. Since $f_{\lambda_1 \cdot \alpha, \lambda_2 \cdot \xi} = \frac{\lambda_1}{\lambda_2} f_{\alpha,\xi}$ and $f_{\lambda \cdot \chi} = \lambda \cdot f_\chi$ for all $\lambda, \lambda_1, \lambda_2 \in \mathbb{F} \setminus \{0\}$, this definition is independent of the choices of χ and ξ.

Using formulas (1), (2) and (3), we can give an explicit description of Φ_k^*, once we have fixed a certain ordered basis $B = (\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n)$ of V. Let $(\omega_1, \omega_2, \dots, \omega_n)$ denote the corresponding dual basis of V^*. For all $i_1, i_2, \dots, i_k \in \{1, \dots, n\}$ satisfying $i_1 < i_2 < \cdots < i_k$, we define

$$\begin{aligned} s(i_1, \dots, i_k) &:= sgn\begin{pmatrix} 1 & \cdots & k & k+1 & \cdots & n \\ i_1 & \cdots & i_k & j_1 & \cdots & j_{n-k} \end{pmatrix} \\ &= (-1)^{k \cdot (n-k)} \cdot sgn\begin{pmatrix} 1 & \cdots & n-k & n-k+1 & \cdots & n \\ j_1 & \cdots & j_{n-k} & i_1 & \cdots & i_k \end{pmatrix}. \end{aligned}$$

As above, let $U = \langle \chi \rangle$ be a one-dimensional subspace of $\bigwedge^k V^*$ and put $\Phi_k^*(U) = \langle \alpha \rangle$. If $\chi = \sum b(i_1, i_2, \dots, i_k) \cdot \omega_{i_1} \wedge \omega_{i_2} \wedge \cdots \wedge \omega_{i_k}$, where the summation ranges over all $i_1, i_2, \dots, i_k \in \{1, \dots, n\}$ satisfying $i_1 < i_2 < \cdots < i_k$, then by equations (1), (2) and (3), α is up to a nonzero factor equal to

$$\sum s(i_1, i_2, \dots, i_k) \cdot b(i_1, i_2, \dots, i_k) \cdot \bar{e}_{j_1} \wedge \bar{e}_{j_2} \wedge \cdots \wedge \bar{e}_{j_{n-k}},$$

where the summation ranges over all numbers $i_1, i_2, \dots, i_k, j_1, j_2, \dots, j_{n-k}$ satisfying $\{i_1, i_2, \dots, i_k, j_1, j_2, \dots, j_{n-k}\} = \{1, 2, \dots, n\}$, $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_{n-k}$.

Definition. We call the vector $\sum s(i_1, i_2, \dots, i_k) \cdot b(i_1, i_2, \dots, i_k) \cdot \bar{e}_{j_1} \wedge \bar{e}_{j_2} \wedge \cdots \wedge \bar{e}_{j_{n-k}} \in \bigwedge^{n-k} V$ the *dual vector* of $\sum b(i_1, i_2, \dots, i_k) \cdot \bar{e}_{i_1} \wedge \bar{e}_{i_2} \wedge \cdots \wedge \bar{e}_{i_k}$ *with respect to* $B = (\bar{e}_1, \dots, \bar{e}_n)$. The following is immediately clear from the above discussion.

Proposition 3.1 *Let B be an ordered basis of V. If $\alpha \in \bigwedge^k V$ and if β denotes the dual vector of α with respect to B, then $\langle\beta\rangle = \Phi_k^*(\langle\rho_{B,k}(\alpha)\rangle)$.*

Proposition 3.2 *Let $B = (\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n)$ be an ordered basis of V.*

(1) Let $\lambda_1, \lambda_2 \in \mathbb{F}$, let $\alpha_1, \alpha_2 \in \bigwedge^k V$ and let β_i, $i \in \{1, 2\}$, be the dual vector of α_i with respect to B. Then $\lambda_1\beta_1 + \lambda_2\beta_2$ is the dual vector of $\lambda_1\alpha_1 + \lambda_2\alpha_2$ with respect to B.

(2) If $\{i_1, \ldots, i_k, j_1, \ldots, j_{n-k}\} = \{1, \ldots, n\}$, then the dual vector of $\bar{e}_{i_1} \wedge \cdots \wedge \bar{e}_{i_k}$ with respect to B is equal to

$$sgn\begin{pmatrix} 1 & \cdots & k & k+1 & \cdots & n \\ i_1 & \cdots & i_k & j_1 & \cdots & j_{n-k} \end{pmatrix} \cdot \bar{e}_{j_1} \wedge \bar{e}_{j_2} \wedge \cdots \wedge \bar{e}_{j_{n-k}}.$$

Proof. (1) This immediately follows from the definition of the notion dual vector.

(2) Let $i'_1, \ldots, i'_k \in \{1, \ldots, n\}$ such that $\{i'_1, \ldots, i'_k\} = \{i_1, \ldots, i_k\}$ and $i'_1 < i'_2 < \cdots < i'_k$. Similarly, let $j'_1, \ldots, j'_{n-k}$ such that $\{j'_1, \ldots, j'_{n-k}\} = \{j_1, \ldots, j_{n-k}\}$ and $j'_1 < j'_2 < \cdots < j'_{n-k}$. Then the dual vector of $\bar{e}_{i_1} \wedge \bar{e}_{i_2} \wedge \cdots \wedge \bar{e}_{i_k} = sgn\begin{pmatrix} i_1 & \cdots & i_k \\ i'_1 & \cdots & i'_k \end{pmatrix} \cdot \bar{e}_{i'_1} \wedge \bar{e}_{i'_2} \wedge \cdots \wedge \bar{e}_{i'_k}$ with respect to B is equal to $sgn\begin{pmatrix} i_1 & \cdots & i_k \\ i'_1 & \cdots & i'_k \end{pmatrix} \cdot s(i'_1, \ldots, i'_k) \cdot \bar{e}_{j'_1} \wedge \bar{e}_{j'_2} \wedge \cdots \wedge \bar{e}_{j'_{n-k}} = sgn\begin{pmatrix} i_1 & \cdots & i_k \\ i'_1 & \cdots & i'_k \end{pmatrix} \cdot s(i'_1, \ldots, i'_k) \cdot sgn\begin{pmatrix} j_1 & \cdots & j_{n-k} \\ j'_1 & \cdots & j'_{n-k} \end{pmatrix} \cdot \bar{e}_{j_1} \wedge \bar{e}_{j_2} \wedge \cdots \wedge \bar{e}_{j_{n-k}} = sgn\begin{pmatrix} 1 & \cdots & k & k+1 & \cdots & n \\ i_1 & \cdots & i_k & j_1 & \cdots & j_{n-k} \end{pmatrix} \cdot \bar{e}_{j_1} \wedge \bar{e}_{j_2} \wedge \cdots \wedge \bar{e}_{j_{n-k}}$. ■

Proposition 3.3 *Let B be an ordered basis of V. If $\beta \in \bigwedge^{n-k} V$ is the dual vector of $\alpha \in \bigwedge^k V$, then $(-1)^{k(n-k)}\alpha$ is the dual vector of β with respect to B.*

Proof. This immediately follows from the definition of the notion dual vector and the fact that $s(i_1, \ldots, i_k) \cdot s(j_1, \ldots, j_{n-k}) = (-1)^{k(n-k)}$ for all $i_1, \ldots, i_k, j_1, \ldots, j_{n-k} \in \mathbb{N}$ satisfying $\{i_1, \ldots, i_k, j_1, \ldots, j_{n-k}\} = \{1, \ldots, n\}$, $i_1 < i_2 < \ldots < i_k$ and $j_1 < j_2 < \cdots < j_{n-k}$. ■

Proposition 3.4 *Let $B = (\bar{e}_1, \ldots, \bar{e}_n)$ be an ordered basis of V. Let $\alpha_1 \in \bigwedge^k V$, $\alpha_2 \in \bigwedge^{n-k} V$, and let β_i, $i \in \{1, 2\}$, denote the dual vector of α_i with respect to B. Then $\alpha_1 \wedge \alpha_2 = \beta_1 \wedge \beta_2$.*

Proof. Let $\alpha_1 = \sum a_1(i_1, \ldots, i_k) \cdot \bar{e}_{i_1} \wedge \cdots \wedge \bar{e}_{i_k}$, where the summation ranges over all $i_1, \ldots, i_k \in \{1, \ldots, n\}$ satisfying $i_1 < \cdots < i_k$. Similarly, put $\alpha_2 = \sum a_2(j_1, \ldots, j_{n-k}) \cdot \bar{e}_{j_1} \wedge \bar{e}_{j_2} \wedge \cdots \wedge \bar{e}_{j_{n-k}}$, where the summation ranges over all $j_1, \ldots, j_{n-k} \in \{1, \ldots, n\}$ satisfying $j_1 < \cdots < j_{n-k}$. Then $\alpha_1 \wedge \alpha_2 = \Big(\sum s(i_1, \ldots, i_k) \cdot a_1(i_1, \ldots, i_k) \cdot a_2(j_1, \ldots, j_{n-k})\Big) \cdot \bar{e}_1 \wedge \cdots \wedge \bar{e}_n$, where the summation ranges over all $i_1, \ldots, i_k, j_1, \ldots, j_{n-k} \in \mathbb{N}$ satisfying $\{i_1, \ldots, i_k, j_1, \ldots, j_{n-k}\} = \{1, 2, \ldots, n\}$, $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_{n-k}$.

Now, $\beta_1 = \sum s(i_1, \ldots, i_k) \cdot a_1(i_1, \ldots, i_k) \cdot \bar{e}_{j_1} \wedge \bar{e}_{j_2} \wedge \cdots \wedge \bar{e}_{j_{n-k}}$ and $\beta_2 = \sum s(j_1, \ldots, j_{n-k}) \cdot a_2(j_1, \ldots, j_{n-k}) \cdot \bar{e}_{i_1} \wedge \bar{e}_{i_2} \wedge \cdots \wedge \bar{e}_{i_k}$, where the summation ranges over all $i_1, \ldots, i_k, j_1, \ldots, j_{n-k} \in \mathbb{N}$ satisfying $\{i_1, \ldots, i_k, j_1, \ldots, j_{n-k}\} = \{1, \ldots, n\}$, $i_1 < \cdots < i_k$ and $j_1 < \cdots <$

j_{n-k}. We find $\beta_1 \wedge \beta_2 = \Big(\sum s(i_1, \ldots, i_k) \cdot a_1(i_1, \ldots, i_k) \cdot a_2(j_1, \ldots, j_{n-k}) \Big) \cdot \bar{e}_1 \wedge \cdots \wedge \bar{e}_n = \alpha_1 \wedge \alpha_2$. ■

Definition. Let $B = (\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n)$ be an ordered basis of V. For every k-dimensional subspace U of V, let $U^{\perp_B}$ denote the $(n-k)$-dimensional subspace of V consisting of all vectors $\bar{v} \in V$ for which $\rho_B(\bar{u})(\bar{v}) = 0$ for all $\bar{u} \in U$. The subspace $U^{\perp_B}$ of V can be defined in an alternative way. Let $(\cdot,\cdot)_B$ denote the following nondegenerate symmetric form on V: $(\sum_{i=1}^n a_i \bar{e}_i, \sum_{i=1}^n b_i \bar{e}_i)_B := \sum_{i=1}^n a_i b_i$. Then $U^{\perp_B}$ is the orthogonal complement of U with respect to the form $(\cdot,\cdot)_B$. Clearly, $(U^{\perp_B})^{\perp_B} = U$.

Proposition 3.5 *Let B be an ordered basis of V. Let $U = \langle \bar{v}_1, \ldots, \bar{v}_k \rangle$ be a k-dimensional subspace of V and put $U^{\perp_B} = \langle \bar{w}_1, \ldots, \bar{w}_{n-k} \rangle$. Then the dual vector of $\bar{v}_1 \wedge \cdots \wedge \bar{v}_k$ with respect to B is proportional to $\bar{w}_1 \wedge \cdots \wedge \bar{w}_{n-k}$.*

Proof. If α denotes the dual vector of $\bar{v}_1 \wedge \cdots \wedge \bar{v}_k$ with respect to B, then $\langle \alpha \rangle = \Phi_k^*(< \rho_B(\bar{v}_1) \wedge \rho_B(\bar{v}_2) \wedge \cdots \wedge \rho_B(\bar{v}_k) >)$ by Proposition 3.1.

Now, extend $(\rho_B(\bar{v}_1), \rho_B(\bar{v}_2), \ldots, \rho_B(\bar{v}_k))$ to an ordered basis $B_1^* = (\rho_B(\bar{v}_1), \ldots, \rho_B(\bar{v}_k), \omega_{k+1}, \ldots, \omega_n)$ of V^* and let $B_1 = (\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n)$ be an ordered basis of V for which B_1^* is the corresponding dual basis. Notice that $\langle \bar{u}_{k+1}, \ldots, \bar{u}_n \rangle = \langle \bar{w}_1, \ldots, \bar{w}_{n-k} \rangle$ and hence $\langle \bar{u}_{k+1} \wedge \cdots \wedge \bar{u}_n \rangle = \langle \bar{w}_1 \wedge \cdots \wedge \bar{w}_{n-k} \rangle$. Using the explicit description of the map Φ_k^* with respect to the ordered bases B_1 and B_1^*, we find $\langle \alpha \rangle = \Phi_k^*(\rho_B(\bar{v}_1) \wedge \rho_B(\bar{v}_2) \wedge \cdots \wedge \rho_B(\bar{v}_k)) = \langle \bar{u}_{k+1} \wedge \cdots \wedge \bar{u}_n \rangle = \langle \bar{w}_1 \wedge \cdots \wedge \bar{w}_{n-k} \rangle$. ■

The following is a corollary of Propositions 3.3, 3.4 and 3.5.

Corollary 3.6 *Let B be an ordered basis of V. Let $\alpha_1 \in \bigwedge^{n-k} V$ and let $\alpha_2 \in \bigwedge^k V$ denote the dual vector of α_1 with respect to B. Let X_1 denote the set of all k-dimensional subspaces $\langle \bar{v}_1, \ldots, \bar{v}_k \rangle$ of V for which $\alpha_1 \wedge \bar{v}_1 \wedge \cdots \wedge \bar{v}_k = 0$. Similarly, let X_2 denote the set of all $(n-k)$-dimensional subspaces $\langle \bar{w}_1, \ldots, \bar{w}_{n-k} \rangle$ of V for which $\alpha_2 \wedge \bar{w}_1 \wedge \cdots \wedge \bar{w}_{n-k} = 0$. Then $X_2 = \{U^{\perp_B} \,|\, U \in X_1\}$.*

Proposition 3.7 *Let B_1 and B_2 be two ordered bases of V and let θ denote the unique element of $GL(V)$ mapping B_1 to B_2, then there exists a $\phi \in GL(V)$ with $\det(\phi) = \det(\theta)^2$ such that the following holds for every $\alpha \in \bigwedge^k V$:*

> *If β_i, $i \in \{1,2\}$, denotes the dual vector of α with respect to B_i, then $\beta_2 = \frac{1}{\det(\theta)} \bigwedge^{n-k}(\phi)(\beta_1)$.*

As a consequence, β_1 and β_2 are semi-equivalent.

Proof. (1) If $B_1 = B_2$, then we can take for ϕ the identical linear transformation of V.

(2) Suppose there exist vectors $\bar{e}_1, \ldots, \bar{e}_n$ of V and a permutation σ of $\{1, \ldots, n\}$ such that $B_1 = (\bar{e}_1, \ldots, \bar{e}_n)$ and $B_2 = (\bar{e}_{\sigma(1)}, \ldots, \bar{e}_{\sigma(n)})$. Let ϕ be the identical transformation of V. Since $\det(\theta) = sgn(\sigma)$, $\det(\phi) = \det(\theta)^2$. If we put α equal to $\sum a(i_1, \ldots, i_k) \cdot \bar{e}_{i_1} \wedge \cdots \wedge \bar{e}_{i_k}$, where Σ denotes the summation over all $i_1, \ldots, i_k, j_1, \ldots, j_{n-k} \in \{1, \ldots, n\}$

satisfying $\{i_1, \dots, i_k, j_1, \dots, j_{n-k}\} = \{1, \dots, n\}$, $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_{n-k}$, then by Proposition 3.2(2), we have

$$\begin{aligned} \beta_1 &= \sum s(i_1, \dots, i_k) \cdot a(i_1, \dots, i_k) \cdot \bar{e}_{j_1} \wedge \cdots \wedge \bar{e}_{j_{n-k}}, \\ \beta_2 &= \sum s(i_1, \dots, i_k) \cdot sgn(\sigma) \cdot a(i_1, \dots, i_k) \cdot \bar{e}_{j_1} \wedge \cdots \wedge \bar{e}_{j_{n-k}}. \end{aligned}$$

So, we have $\beta_2 = sgn(\sigma) \cdot \beta_1 = \frac{1}{\det(\theta)} \bigwedge^{n-k}(\phi)(\beta_1)$.

(3) Suppose there exist vectors $\bar{e}_1, \dots, \bar{e}_n$ of V and a $\lambda \in \mathbb{F} \setminus \{0\}$ such that $B_1 = (\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n)$ and $B_2 = (\lambda\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n)$. Let ϕ be the following element of $GL(V)$: $\bar{e}_1 \mapsto \lambda^2\bar{e}_1$; $\bar{e}_i \mapsto \bar{e}_i$, $\forall i \in \{2, \dots, n\}$. Then $\det(\phi) = \lambda^2 = \det(\theta)^2$. Put $\alpha = \sum_1 a(i_1, \dots, i_k) \cdot \bar{e}_{i_1} \wedge \cdots \wedge \bar{e}_{i_k} + \sum_2 a(i_1, \dots, i_k) \cdot \bar{e}_{i_1} \wedge \cdots \wedge \bar{e}_{i_k} = \sum_1 \frac{a(i_1,\dots,i_k)}{\lambda} \cdot (\lambda \cdot \bar{e}_{i_1}) \wedge \bar{e}_{i_2} \wedge \cdots \wedge \bar{e}_{i_k} + \sum_2 a(i_1, \dots, i_k) \cdot \bar{e}_{i_1} \wedge \cdots \wedge \bar{e}_{i_k}$, where

• Σ_1 denotes the summation ranging over all $i_1, \dots, i_k, j_1, \dots, j_{n-k} \in \{1, \dots, n\}$ satisfying $\{i_1, \dots, i_k, j_1, \dots, j_{n-k}\} = \{1, \dots, n\}$, $1 = i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_{n-k}$;

• Σ_2 denotes the summation ranging over all $i_1, \dots, i_k, j_1, \dots, j_{n-k} \in \{1, \dots, n\}$ satisfying $\{i_1, \dots, i_k, j_1, \dots, j_{n-k}\} = \{1, \dots, n\}$, $i_1 < i_2 < \cdots < i_k$ and $1 = j_1 < j_2 < \cdots < j_{n-k}$.

We have

$$\begin{aligned} \beta_1 = &\sum_1 s(i_1, \dots, i_k) \cdot a(i_1, \dots, i_k) \cdot \bar{e}_{j_1} \wedge \cdots \wedge \bar{e}_{j_{n-k}} \\ &+ \sum_2 s(i_1, \dots, i_k) \cdot a(i_1, \dots, i_k) \cdot \bar{e}_{j_1} \wedge \cdots \wedge \bar{e}_{j_{n-k}}, \end{aligned}$$

and

$$\begin{aligned} \beta_2 = &\sum_1 \frac{1}{\lambda} s(i_1, \dots, i_k) \cdot a(i_1, \dots, i_k) \cdot \bar{e}_{j_1} \wedge \cdots \wedge \bar{e}_{j_{n-k}} \\ &+ \sum_2 \lambda \cdot s(i_1, \dots, i_k) \cdot a(i_1, \dots, i_k) \cdot \bar{e}_{j_1} \wedge \cdots \wedge \bar{e}_{j_{n-k}}. \end{aligned}$$

Hence, $\beta_2 = \frac{1}{\lambda} \bigwedge^{n-k}(\phi)(\beta_1) = \frac{1}{\det(\theta)} \bigwedge^{n-k}(\phi)(\beta_1)$.

(4) Suppose there exist vectors $\bar{e}_1, \dots, \bar{e}_n$ of V such that $B_1 = (\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n)$ and $B_2 = (\bar{e}_1 + \bar{e}_2, \bar{e}_2, \dots, \bar{e}_n)$. Let ϕ be the following map of $GL(V)$: $\bar{e}_1 \mapsto \bar{e}_1 + \bar{e}_2$; $\bar{e}_2 \mapsto -\bar{e}_1$; $\bar{e}_i \mapsto \bar{e}_i$, $\forall i \in \{3, \dots, n\}$. Then $\det(\phi) = 1 = \det(\theta)^2$. Put $\alpha = \sum_1 a(i_1, \dots, i_k) \cdot \bar{e}_{i_1} \wedge \cdots \wedge \bar{e}_{i_k} + \sum_2 a(i_1, \dots, i_k) \cdot \bar{e}_{i_1} \wedge \cdots \wedge \bar{e}_{i_k} + \sum_3 a(i_1, \dots, i_k) \cdot \bar{e}_{i_1} \wedge \cdots \wedge \bar{e}_{i_k} + \sum_4 a(i_1, \dots, i_k) \cdot \bar{e}_{i_1} \wedge \cdots \wedge \bar{e}_{i_k}$, where Σ_l, $l \in \{1, 2, 3, 4\}$, denotes the summation ranging over all $i_1, \dots, i_k, j_1, \dots, j_{n-k} \in \{1, \dots, n\}$ satisfying $\{i_1, \dots, i_k, j_1, \dots, j_{n-k}\} = \{1, \dots, n\}$, $i_1 < i_2 < \cdots < i_k$, $j_1 < j_2 < \cdots < j_{n-k}$ and Property (P_l). Here:

$$\begin{aligned} &(P_1): \ i_1 = 1, \ i_2 = 2; \qquad && (P_2): \ i_1 = 1, \ j_1 = 2; \\ &(P_3): \ i_1 = 2, \ j_1 = 1; \qquad && (P_4): \ j_1 = 1, \ j_2 = 2. \end{aligned}$$

In the sum Σ_1, we have $\bar{e}_{i_1} \wedge \cdots \wedge \bar{e}_{i_k} = (\bar{e}_1 + \bar{e}_2) \wedge \bar{e}_2 \wedge \bar{e}_{i_3} \wedge \cdots \wedge \bar{e}_{i_k}$. In the sum Σ_2, we have $\bar{e}_{i_1} \wedge \cdots \wedge \bar{e}_{i_k} = (\bar{e}_1 + \bar{e}_2) \wedge \bar{e}_{i_2} \wedge \cdots \wedge \bar{e}_{i_k} - \bar{e}_2 \wedge \bar{e}_{i_2} \wedge \cdots \bar{e}_{i_k}$. In the sum Σ_3, we have $\bar{e}_{i_1} \wedge \cdots \wedge \bar{e}_{i_k} = \bar{e}_2 \wedge \bar{e}_{i_2} \wedge \cdots \wedge \bar{e}_{i_k}$. We have

$$\begin{aligned}\beta_1 \quad = \quad & \sum_1 s(i_1, \ldots, i_k) \cdot a(i_1, \ldots, i_k) \cdot \bar{e}_{j_1} \wedge \cdots \wedge \bar{e}_{j_{n-k}} \\ & + \sum_2 s(i_1, \ldots, i_k) \cdot a(i_1, \ldots, i_k) \cdot \bar{e}_{j_1} \wedge \cdots \wedge \bar{e}_{j_{n-k}} \\ & + \sum_3 s(i_1, \ldots, i_k) \cdot a(i_1, \ldots, i_k) \cdot \bar{e}_{j_1} \wedge \cdots \wedge \bar{e}_{j_{n-k}} \\ & + \sum_4 s(i_1, \ldots, i_k) \cdot a(i_1, \ldots, i_k) \cdot \bar{e}_{j_1} \wedge \cdots \wedge \bar{e}_{j_{n-k}},\end{aligned}$$

and

$$\begin{aligned}\beta_2 \quad = \quad & \sum_1 s(i_1, \ldots, i_k) \cdot a(i_1, \ldots, i_k) \cdot \bar{e}_{j_1} \wedge \cdots \wedge \bar{e}_{j_{n-k}} \\ & + \sum_2 s(i_1, \ldots, i_k) \cdot a(i_1, \ldots, i_k) \cdot (\bar{e}_2 - (\bar{e}_1 + \bar{e}_2)) \wedge \bar{e}_{j_2} \wedge \cdots \wedge \bar{e}_{j_{n-k}} \\ & + \sum_3 s(i_1, \ldots, i_k) \cdot a(i_1, \ldots, i_k) \cdot (\bar{e}_1 + \bar{e}_2) \wedge \bar{e}_{j_2} \wedge \cdots \wedge \bar{e}_{j_{n-k}} \\ & + \sum_4 s(i_1, \ldots, i_k) \cdot a(i_1, \ldots, i_k) \cdot (\bar{e}_1 + \bar{e}_2) \wedge \bar{e}_2 \wedge \bar{e}_{j_3} \wedge \cdots \wedge \bar{e}_{j_{n-k}}.\end{aligned}$$

One readily verifies that $\beta_2 = \bigwedge^{n-k}(\phi)(\beta_1) = \frac{1}{\det(\theta)} \bigwedge^{n-k}(\phi)(\beta_1)$.

(5) Suppose now that B_1 and B_2 are two arbitrary distinct ordered bases of V. Then there exist ordered bases $C_0, C_1, \ldots, C_k$ of V such that: (i) $C_0 = B_1$; (ii) $C_k = B_2$; (iii) C_i, $i \in \{1, \ldots, k\}$, is related to C_{i-1} as described in (1), (2), (3) or (4) above. So, it suffices to prove that if the proposition holds for pairs (C_0, C_1) and (C_1, C_2) of ordered bases of V, then the proposition also holds for the pair (C_0, C_2). Let θ_i, $i \in \{1, 2\}$, denote the unique element of $GL(V)$ mapping C_{i-1} to C_i, and let ϕ_i denote an element of $GL(V)$ associated with the pair (C_{i-1}, C_i, θ_i). So, $\det(\phi_1) = \det(\theta_1)^2$, $\det(\phi_2) = \det(\theta_2)^2$. Moreover, if $\alpha \in \bigwedge^k V$ and if β_i, $i \in \{0, 1, 2\}$, denotes the dual vector of α with respect to C_i, then $\beta_1 = \frac{1}{\det(\theta_1)} \bigwedge^{n-k}(\phi_1)(\beta_0)$ and $\beta_2 = \frac{1}{\det(\theta_2)} \bigwedge^{n-k}(\phi_2)(\beta_1)$. It follows that $\beta_2 = \frac{1}{\det(\theta_2 \circ \theta_1)} \bigwedge^{n-k}(\phi_2 \circ \phi_1)(\beta_0)$. Here, $\theta_2 \circ \theta_1$ is the unique element of $GL(V)$ mapping C_0 to C_2, and $\det(\phi_2 \circ \phi_1) = \det(\phi_2) \cdot \det(\phi_1) = \det(\theta_2)^2 \cdot \det(\theta_1)^2 = \det(\theta_2 \circ \theta_1)^2$. ∎

Proposition 3.8 *Let B be an ordered basis of V. Let α_1 and α_2 be two vectors of $\bigwedge^k V$ and let β_i, $i \in \{1, 2\}$, denote the dual vector of α_i with respect to B. Then α_1 and α_2 are semi-equivalent if and only if β_1 and β_2 are semi-equivalent.*

Proof. We give two distinct proofs.

(1) Clearly, α_1 and α_2 are semi-equivalent if and only if $\chi_1 := \rho_{B,k}(\alpha_1)$ and $\chi_2 := \rho_{B,k}(\alpha_2)$ are semi-equivalent. By Proposition 2.1(1), χ_1 and χ_2 are semi-equivalent if

and only if f_{χ_1} and f_{χ_2} are semi-equivalent. Notice that if ξ is an arbitrary nonzero vector of $\bigwedge^n V$, then $f_{\beta_i,\xi}$, $i \in \{1,2\}$, is a nonzero multiple of f_{χ_i} by Proposition 3.1. So, by Proposition 2.1(3), f_{χ_1} and f_{χ_2} are semi-equivalent if and only if β_1 and β_2 are semi-equivalent. The proposition follows.

(2) In view of Proposition 3.3, it suffices to prove the "only if" part of the proposition. Suppose α_1 and α_2 are semi-equivalent. Then there exists a $\lambda \in \mathbb{F} \setminus \{0\}$ and a $\theta \in GL(V)$ such that $\lambda \cdot \alpha_2 = \bigwedge^k(\theta)(\alpha_1)$. Put $B = (\bar{e}_1, \ldots, \bar{e}_n)$ and $B' = (\theta(\bar{e}_1), \ldots, \theta(\bar{e}_n))$. Put $\alpha_1 = \sum a(i_1, \ldots, i_k) \cdot \bar{e}_{i_1} \wedge \cdots \wedge \bar{e}_{i_k}$, where the summation Σ ranges over all $i_1, \ldots, i_k \in \{1, \ldots, n\}$ satisfying $i_1 < i_2 < \cdots < i_k$. Then $\lambda \cdot \alpha_2 = \sum a(i_1, \ldots, i_k) \cdot \theta(\bar{e}_{i_1}) \wedge \cdots \wedge \theta(\bar{e}_{i_k})$. Obviously, $\bigwedge^{n-k}(\theta)(\beta_1)$ is the dual vector of $\lambda \cdot \alpha_2$ with respect to the basis B'. So, β_1 is semi-equivalent with the dual vector of α_2 with respect to B'. By Proposition 3.7, β_1 is also semi-equivalent with the dual vector β_2 of α_2 with respect to B. ∎

Corollary 3.9 *Let B_1, B_2 be two ordered bases of V, and let α_1, α_2 be two vectors of $\bigwedge^k V$. Let β_i, $i \in \{1,2\}$, denote the dual vector of α_i with respect to B_i. Then α_1 and α_2 are semi-equivalent if and only if β_1 and β_2 are semi-equivalent.*

Proof. Let β_2' denote the dual vector of α_2 with respect to B_1. Then β_2 and β_2' are semi-equivalent by Proposition 3.7. Now, by Proposition 3.8, α_1 and α_2 are semi-equivalent if and only if β_1 and β_2' are semi-equivalent, i.e., if and only if β_1 and β_2 are semi-equivalent. ∎

4 Bivectors and $(n-2)$-vectors

In view of the connection which exists between the alternating bilinear forms on a vector space V and the bivectors of the dual space V^* of V, the classification, up to equivalence, of the bivectors of V (or equivalently, of V^*), is well-known and readily obtained. We will discuss this in Section 4.1 where we will also take the opportunity to derive a property of bivectors (Proposition 4.1(2)) which we will need later. The classification of the $(n-2)$-vectors of V is discussed in Section 4.2. We found no suitable reference for this latter classification in the literature.

4.1 Bivectors

Let V be a vector space of dimension $n \geq 0$ over a field $\mathbb{F}$. The alternating bilinear forms on V are also called the *symplectic forms* on V. The radical $Rad(f)$ of a symplectic form $f : V \times V \to \mathbb{F}$ is the set of all $\bar{v} \in V$ such that $f(\bar{v}, \bar{w}) = 0$, $\forall \bar{w} \in V$.

Suppose $Rad(f) = 0$. Then the symplectic form f is nondegenerate and $n = 2m$ is even. An ordered basis $(\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_m, \bar{f}_m)$ of V is then called a *hyperbolic basis* of V (with respect to f) if $f(\bar{e}_i, \bar{e}_j) = f(\bar{f}_i, \bar{f}_j) = 0$ and $f(\bar{e}_i, \bar{f}_j) = \delta_{ij}$ for all $i, j \in \{1, \ldots, m\}$.

In the general case, $2m := n - \dim(Rad(f))$ is even. Let $\{\bar{g}_{2m+1}, \ldots, \bar{g}_n\}$ be a basis of $Rad(f)$. If U is a subspace of V complementary to $Rad(f)$, then the form f_U induced by f on U is a nondegenerate symplectic form. If $(\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_m, \bar{f}_m)$ denotes a hyperbolic basis of U with respect to f_U, then f is completely determined by $(\bar{e}_1, \bar{f}_1, \dots, \bar{e}_m, \bar{f}_m)$ and $(\bar{g}_{2m+1}, \dots, \bar{g}_n)$. So, for every $m \in \{0, \dots, \lfloor \frac{n}{2} \rfloor\}$, there exists, up to equivalence, a unique nondegenerate symplectic form f for which $Rad(f)$ has dimension $n - 2m$. So, there are up to equivalence precisely $\lfloor \frac{n}{2} \rfloor + 1$ symplectic forms on V.

As mentioned in Section 2.2, there exists a one-to-one correspondence between the symplectic forms on V and the elements of $\bigwedge^2 V^*$, where V^* is the dual space of V. If $(\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n)$ is an ordered basis of V and $(\omega_1, \omega_2, \dots, \omega_n)$ denotes the corresponding dual basis of V^*, then the $\lfloor \frac{n}{2} \rfloor + 1$ nonequivalent symplectic forms on V correspond to the bivectors $\sum_{i=1}^{k} \omega_{2i-1} \wedge \omega_{2i}$, $k \in \{0, \dots, \lfloor \frac{n}{2} \rfloor\}$, of V^*. If f is the symplectic form on V associated to $\sum_{i=1}^{k} \omega_{2i-1} \wedge \omega_{2i}$, then $Rad(f) = \langle \bar{e}_{2k+1}, \bar{e}_{2k+2}, \dots, \bar{e}_n \rangle$. If $n = 2m$ is even and if f is the symplectic form on V corresponding to $\sum_{i=1}^{m} \omega_{2i-1} \wedge \omega_{2i}$, then $(\bar{e}_1, \bar{e}_2, \dots, \bar{e}_{2m})$ is a hyperbolic basis of V with respect to f.

Now, suppose that $\dim(V) = n = 2m \geq 2$ is even and that f is a nondegenerate symplectic form on V. Then the element of $\bigwedge^2 V^*$ corresponding to f can we written in the form $\sum_{i=1}^{m} \omega_{2i-1} \wedge \omega_{2i}$, where $\omega_1, \omega_2, \dots, \omega_{2m}$ are linearly independent elements of V^*. Let ω_1' and ω_2' be two linearly independent elements of V^*. Let U denote the $(n-2)$-dimensional subspace of V consisting of all vectors $\bar{u} \in V$ for which $\omega_1'(\bar{u}) = \omega_2'(\bar{u}) = 0$ and let f_U denote the alternating bilinear form on U induced by f. Then $Rad(f_U)$ is even and has dimension at most 2. We distinguish two cases.

(1) $Rad(f_U) = \{\bar{o}\}$. Let $(\bar{e}_2, \bar{f}_2, \dots, \bar{e}_m, \bar{f}_m)$ be a hyperbolic basis of U with respect to f_U. This hyperbolic basis can be extended to a hyperbolic basis $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \dots, \bar{e}_m, \bar{f}_m)$ of V (with respect to f) whose corresponding dual basis of V^* is of the form $(\omega_1', \lambda \cdot \omega_2', \omega_3', \dots, \omega_{2m}')$ where $\lambda \in \mathbb{F} \setminus \{0\}$. It follows that $\omega_1 \wedge \omega_2 + \cdots + \omega_{2m-1} \wedge \omega_{2m} = \omega_1' \wedge (\lambda \cdot \omega_2') + \omega_3' \wedge \omega_4' + \cdots + \omega_{2m-1}' \wedge \omega_{2m}'$ since $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \dots, \bar{e}_m, \bar{f}_m)$ is a hyperbolic basis of the symplectic forms determined by $\omega_1 \wedge \omega_2 + \cdots + \omega_{2m-1} \wedge \omega_{2m}$ and $\omega_1' \wedge (\lambda \cdot \omega_2') + \omega_3' \wedge \omega_4' + \cdots + \omega_{2m-1}' \wedge \omega_{2m}'$.

(2) $Rad(f_U)$ has dimension 2. Let W denote a subspace of U complementary to $Rad(f_U)$. Let $(\bar{f}_1, \bar{f}_2)$ be a basis of $Rad(f_U)$ and let $(\bar{e}_3, \bar{f}_3, \dots, \bar{e}_m, \bar{f}_m)$ be a hyperbolic basis of W with respect to the form f_W induced by f on W. We can extend $(\bar{f}_1, \bar{f}_2, \bar{e}_3, \bar{f}_3, \dots, \bar{e}_m, \bar{f}_m)$ to a hyperbolic basis $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \dots, \bar{e}_m, \bar{f}_m)$ of V whose associated dual basis of V^* is of the form $(\omega_1', \omega_3', \omega_2', \omega_4', \omega_5', \dots, \omega_{2m}')$. It follows that $\omega_1 \wedge \omega_2 + \cdots + \omega_{2m-1} \wedge \omega_{2m}$ is equal to $\omega_1' \wedge \omega_3' + \omega_2' \wedge \omega_4' + \omega_5' \wedge \omega_6' + \cdots + \omega_{2m-1}' \wedge \omega_{2m}'$ since the two symplectic forms associated with these vectors of $\bigwedge^2 V^*$ have $(\bar{e}_1, \bar{f}_1, \dots, \bar{e}_m, \bar{f}_m)$ as a hyperbolic basis.

We can conclude:

Proposition 4.1 *Let V be a vector space of dimension $n \geq 2$ over a field $\mathbb{F}$ and let $\{\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n\}$ be a basis of V.*

(1) *There are $\lfloor \frac{n}{2} \rfloor$ equivalence classes of nonzero bivectors of V. The $\lfloor \frac{n}{2} \rfloor$ bivectors $\sum_{i=1}^{k} \bar{e}_{2i-1} \wedge \bar{e}_{2i}$, $k \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$, are representatives of these $\lfloor \frac{n}{2} \rfloor$ classes.*

(2) *Let $k \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$ and let $\bar{e}_1', \bar{e}_2'$ be two linearly independent vectors of $\langle \bar{e}_1, \dots, \bar{e}_{2k} \rangle$. Then there exist vectors $\bar{e}_3', \dots, \bar{e}_{2k}'$ such that $\sum_{i=1}^{k} \bar{e}_{2i-1} \wedge \bar{e}_{2i}$ is equal to either*

$\bar{e}'_1 \wedge \bar{e}'_3 + \bar{e}'_2 \wedge \bar{e}'_4 + \sum_{i=3}^{k} \bar{e}'_{2i-1} \wedge \bar{e}'_{2i}$ *(only if $k \geq 2$) or* $\bar{e}'_1 \wedge (\lambda \bar{e}'_2) + \sum_{i=2}^{k} \bar{e}'_{2i-1} \wedge \bar{e}'_{2i}$ *for some* $\lambda \in \mathbb{F} \setminus \{0\}$.

Notice that if $k \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$ and $\lambda \in \mathbb{F} \setminus \{0\}$, then the vectors $\sum_{i=1}^{k} \bar{e}_{2i-1} \wedge \bar{e}_{2i}$ and $\lambda \cdot \left(\sum_{i=1}^{k} \bar{e}_{2i-1} \wedge \bar{e}_{2i} \right)$ are equivalent. (Consider the element of $GL(V)$ mapping $\bar{e}_i$ to $\bar{e}_i$ if $i \in \{1, \ldots, n\}$ is odd and $\bar{e}_i$ to $\lambda \cdot \bar{e}_i$ if $i \in \{1, \ldots, n\}$ is even). So, up to semi-equivalence, there are also $\lfloor \frac{n}{2} \rfloor$ nonzero bivectors.

4.2 $(n-2)$-vectors

Suppose V is a vector space of dimension $n \geq 3$ over a field $\mathbb{F}$. Let $B = (\bar{e}_1, \ldots, \bar{e}_n)$ be an ordered basis of V. Recall that up to semi-equivalence there are precisely $\lfloor \frac{n}{2} \rfloor$ nonzero bivectors of V, namely the $\lfloor \frac{n}{2} \rfloor$ bivectors $\alpha_k = \sum_{i=1}^{k} \bar{e}_{2i-1} \wedge \bar{e}_{2i}$, $k \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$. Let β_i, $i \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$, denote the dual vector of α_i with respect to the basis B. By Proposition 3.8, there are up to semi-equivalence $\lfloor \frac{n}{2} \rfloor$ nonzero $(n-2)$-vectors of V, namely the $(n-2)$-vectors β_k, $k \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$.

Proposition 4.2 (1) *Let $\lambda \in \mathbb{F} \setminus \{0\}$ and $k \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$ such that $2k \neq n$. Then β_k and $\lambda \cdot \beta_k$ are equivalent.*

(2) *Let $\lambda \in \mathbb{F} \setminus \{0\}$ and $n = 2m$ even. Then β_m and $\lambda \cdot \beta_m$ are equivalent if and only if there exists a $\mu \in \mathbb{F}$ such that $\lambda = \mu^{m-1}$.*

Proof. (1) Let θ be the map of $GL(V)$ mapping $\bar{e}_i$ to $\bar{e}_i$ if $i \in \{1, \ldots, n-1\}$ and $\bar{e}_i$ to $\lambda \cdot \bar{e}_i$ if $i = n$. Then $\bigwedge^{n-2}(\theta)(\beta_k) = \lambda \cdot \beta_k$.

(2) Suppose there exists a $\mu \in \mathbb{F}$ such that $\lambda = \mu^{m-1}$. Then let θ be the map of $GL(V)$ mapping $\bar{e}_i$ to $\bar{e}_i$ if $i \in \{1, \ldots, n\}$ is odd and $\bar{e}_i$ to $\mu \cdot \bar{e}_i$ if $i \in \{1, \ldots, n\}$ is even. Then $\bigwedge^{m-2}(\theta)(\beta_m) = \lambda \cdot \beta_m$.

Conversely, suppose that β_m and $\lambda \cdot \beta_m$ are equivalent. Let θ be a map of $GL(V)$ such that $\bigwedge^{m-2}(\theta)(\beta_m) = \lambda \cdot \beta_m$. Put $\xi = \bar{e}_1 \wedge \bar{e}_2 \wedge \cdots \wedge \bar{e}_n$. Then $f := f_{\beta_m, \xi}$ is a symplectic form on V. For all $\bar{v}_1, \bar{v}_2 \in V$, we have

$$\begin{aligned}
\beta_m \wedge \bar{v}_1 \wedge \bar{v}_2 &= f(\bar{v}_1, \bar{v}_2) \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \cdots \wedge \bar{e}_n, \\
\wedge^{m-2}(\theta)(\beta_m) \wedge \theta(\bar{v}_1) \wedge \theta(\bar{v}_2) &= f(\bar{v}_1, \bar{v}_2) \cdot \det(\theta) \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \cdots \wedge \bar{e}_n, \\
\lambda \cdot \beta_m \wedge \theta(\bar{v}_1) \wedge \theta(\bar{v}_2) &= f(\bar{v}_1, \bar{v}_2) \cdot \det(\theta) \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \cdots \wedge \bar{e}_n, \\
\lambda \cdot f(\theta(\bar{v}_1), \theta(\bar{v}_2)) &= \det(\theta) \cdot f(\bar{v}_1, \bar{v}_2).
\end{aligned}$$

Now, let $\mathbb{F}'$ be a (possibly trivial) algebraic extension of $\mathbb{F}$ containing a square root δ of $\frac{\det(\theta)}{\lambda}$. Let V' be an n-dimensional vector space over $\mathbb{F}'$ which also has $\{\bar{e}_1, \ldots, \bar{e}_n\}$ as a basis. Then f induces a symplectic form f' on V' and θ induces an element θ' of $GL(V')$. Put $\theta'' := \frac{1}{\delta}\theta'$. Then $f'(\theta''(\bar{v}'_1), \theta''(\bar{v}'_2)) = f'(\bar{v}'_1, \bar{v}'_2)$ for all $\bar{v}'_1, \bar{v}'_2 \in V'$. This implies that θ'' belongs to the symplectic group $Sp(V', f')$. So, $1 = \det(\theta'') = \frac{1}{\delta^{2m}} \det(\theta') = \frac{\lambda^m}{\det(\theta)^m} \det(\theta)$. Hence, $\lambda = \left(\frac{\det(\theta)}{\lambda} \right)^{m-1}$. ∎

Corollary 4.3 (1) *If $n \geq 3$ is odd, then up to equivalence, there are $\lfloor \frac{n}{2} \rfloor$ nonzero $(n-2)$-vectors of V.*

(2) *If $n = 2m \geq 4$ is even, then up to equivalence, there are $\lfloor \frac{n-2}{2} \rfloor + [\mathbb{F}^* : G]$ nonzero $(n-2)$-vectors of V. Here, $\mathbb{F}^*$ denotes the multiplicative group of the field $\mathbb{F}$ and G denotes the subgroup of $\mathbb{F}^*$ consisting of all $(m-1)$-th powers.*

Example. Suppose V is a vector space of dimension $2m \geq 6$ over the field $\mathbb{Q}$ of the rational numbers. Then there are infinitely many nonequivalent nonzero $(n-2)$-vectors, while there are only $\lfloor \frac{n}{2} \rfloor$ nonequivalent nonzero bivectors.

5 The isomorphism problem for hyperplanes of projective Grassmannians

Let V be a vector space of dimension $n \geq 2$ over a field $\mathbb{F}$, let $k \in \{1, \ldots, n-1\}$ and let $A_{n-1,k}(\mathbb{F})$ be the Grassmannian of the $(k-1)$-dimensional subspaces of $\mathrm{PG}(V)$.

Suppose H_1 and H_2 are two hyperplanes of $\mathrm{PG}(V)$ and that $\alpha_i \in \bigwedge^{n-k} V$, $i \in \{1,2\}$, is a representative vector of H_i. The following problem can then be posed.

> $(*)$ What relationship exists between α_1 and α_2 if the hyperplanes H_1 and H_2 are isomorphic?

In order to give an answer to Problem $(*)$, we must first know the full group of automorphisms of $A_{n-1,k}(\mathbb{F})$. This group was determined by Chow [5].

Proposition 5.1 ([5]) (1) *If $n \neq 2k$, then every automorphism of $A_{n-1,k}(\mathbb{F})$ is induced by a collineation of* $\mathrm{PG}(V)$.

(2) *If $n = 2k$, then every automorphism of $A_{n-1,k}(\mathbb{F})$ is induced by a collineation or a duality of* $\mathrm{PG}(V)$.

The following proposition deals with the case of automorphisms which are induced by a projectivity of $\mathrm{PG}(V)$.

Proposition 5.2 *Let H_1 and H_2 be two hyperplanes of $A_{n-1,k}(\mathbb{F})$ and let α_i, $i \in \{1,2\}$, be a representative vector of H_i. Then there is an automorphism of $A_{n-1,k}(\mathbb{F})$ induced by a projectivity of* $\mathrm{PG}(V)$ *mapping H_1 to H_2 if and only if the $(n-k)$-vectors α_1 and α_2 are semi-equivalent.*

Proof. If $\theta \in GL(V)$, then clearly

$$\wedge^{n-k}(\theta)(\alpha_1) \wedge \theta(\bar{v}_1) \wedge \cdots \wedge \theta(\bar{v}_k) = \det(\theta) \cdot (\alpha_1 \wedge \bar{v}_1 \wedge \cdots \wedge \bar{v}_k). \qquad (4)$$

(1) Suppose α_1 and α_2 are semi-equivalent. Then there exists a $\theta \in GL(V)$ such that $\bigwedge^{n-k}(\theta)(\alpha_1)$ and α_2 are proportional. Then (4) implies that $H_2 = \{\pi^\eta \,|\, \pi \in H_1\}$, where η is the projectivity of $\mathrm{PG}(V)$ induced by θ.

(2) Suppose there exists a $\theta \in GL(V)$ such that $H_2 = H_1^\eta$, where η is the projectivity of PG(V) induced by θ. By (4), $\bigwedge^{n-k}(\theta)(\alpha_1)$ is a representative vector of H_2. So, $\bigwedge^{n-k}(\theta)(\alpha_1)$ is proportional to α_2, and α_1 and α_2 are semi-equivalent. ∎

The "if" part of Proposition 5.2 can be generalized.

Proposition 5.3 *Let $l \in \{1, \ldots, k\}$ and let α_1, α_2 be two nonzero $(n-k)$-vectors. Let X_i, $i \in \{1, 2\}$, denote the set of all $(l-1)$-dimensional subspaces $\langle \bar{v}_1, \ldots, \bar{v}_l \rangle$ of* PG(V) *such that $\alpha_i \wedge \bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_l = 0$. If α_1 and α_2 are semi-equivalent, then there exists a projectivity of* PG(V) *mapping X_1 to X_2.*

Proof. Let $\theta \in GL(V)$ such that the vectors $\bigwedge^{n-k}(\theta)(\alpha_1)$ and α_2 are proportional. For l linearly independent vectors $\bar{v}_1, \ldots, \bar{v}_l$ of V, we have $\langle \bar{v}_1, \ldots, \bar{v}_l \rangle \in X_1 \Leftrightarrow \alpha_1 \wedge \bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_l = 0 \Leftrightarrow \bigwedge^{n-k+l}(\theta)(\alpha_1 \wedge \bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_l) = 0 \Leftrightarrow \bigwedge^{n-k}(\theta)(\alpha_1) \wedge \theta(\bar{v}_1) \wedge \theta(\bar{v}_2) \wedge \cdots \wedge \theta(\bar{v}_l) = 0 \Leftrightarrow \alpha_2 \wedge \theta(\bar{v}_1) \wedge \cdots \wedge \theta(\bar{v}_2) = 0$. So, if η denotes the projectivity of PG(V) associated to θ, then $X_1^\eta = X_2$. ∎

Now, suppose $B = (\bar{e}_1, \ldots, \bar{e}_n)$ is a given ordered basis of V. If ψ is an automorphism of $\mathbb{F}$, then we define:

(i) $\alpha^{\psi_B} = \sum a(i_1, \ldots, i_l)^\psi \cdot \bar{e}_{i_1} \wedge \bar{e}_{i_2} \wedge \cdots \wedge \bar{e}_{i_l}$ for every l-vector $\alpha = \sum a(i_1, \ldots, i_l) \cdot \bar{e}_{i_1} \wedge \bar{e}_{i_2} \wedge \cdots \wedge \bar{e}_{i_l}$ of V. Here, $l \in \{1, \ldots, n\}$ and the summation Σ ranges over all $i_1, i_2, \ldots, i_l \in \{1, \ldots, n\}$ satisfying $i_1 < i_2 < \cdots < i_l$.

(ii) $p^{\psi_B} = \langle \bar{x}^{\psi_B} \rangle$ for every point $p = \langle \bar{x} \rangle$ of PG(V).

(iii) $\pi^{\psi_B} := \{p^{\psi_B} \mid p \in \pi\}$ for every subspace π of PG(V).

So, ψ_B has different meanings. In (ii) and (iii), ψ_B is regarded as a collineation of PG(V).

If B is some fixed ordered basis of V, then every collineation of PG(V) is of the form $\eta \circ \psi_B$, where η is some projectivity of PG(V) and ψ is some automorphism of $\mathbb{F}$. So, the following proposition in combination with Proposition 5.2 basically gives an answer to Problem $(*)$ if there exists an automorphism arising from a collineation of PG(V) which maps H_1 to H_2.

Proposition 5.4 *Let B be an ordered basis of V and let ψ be an automorphism of $\mathbb{F}$. Suppose α is a representative vector of a hyperplane H of $A_{n-1,k}(\mathbb{F})$. Then α^{ψ_B} is a representative vector of the hyperplane $H^{\psi_B} := \{\pi^{\psi_B} \mid \pi \in H\}$ of $A_{n-1,k}(\mathbb{F})$.*

Proof. This immediately follows from the fact that $(\alpha_1 \wedge \alpha_2)^{\psi_B} = \alpha_1^{\psi_B} \wedge \alpha_2^{\psi_B}$ for all $\alpha_1 \in \bigwedge^{n-k} V$ and all $\alpha_2 \in \bigwedge^k V$. ∎

Again, suppose that $B = (\bar{e}_1, \ldots, \bar{e}_n)$ is an ordered basis of V. Then the permutation of the set of subspaces of V defined by $U \mapsto U^{\perp_B}$ induces a duality ν_B of PG(V). The following is an immediate consequence of Corollary 3.6.

Proposition 5.5 *Let B be an ordered basis of V and let α be a representative vector of a hyperplane H of $A_{n-1,k}(\mathbb{F})$. Then the dual vector of α with respect to B is a representative vector of the hyperplane H^{ν_B} of $A_{n-1,n-k}(\mathbb{F})$.*

In the special case $n = 2k$, the group of automorphisms of $A_{n-1,k}(\mathbb{F})$ induced by collineations of $\mathrm{PG}(V)$ is a (normal) subgroup of index 2 of the full group of automorphisms of $A_{n-1,k}(\mathbb{F})$. So, Propositions 5.2, 5.4 and 5.5 basically give a complete answer to Problem $(*)$ if $n = 2k$.

Using the results of Section 4, one can now easily verify that there are up to isomorphism $\lfloor \frac{n}{2} \rfloor$ hyperplanes of $A_{n-1,2}(\mathbb{F})$ and $\lfloor \frac{n}{2} \rfloor$ hyperplanes in $A_{n-1,n-2}(\mathbb{F})$. Moreover, if H_1 and H_2 are two isomorphic hyperplanes of $A_{3,2}(\mathbb{F})$, then there exists an automorphism of $A_{3,2}(\mathbb{F})$ induced by a collineation of the ambient projective space $\mathrm{PG}(3,\mathbb{F})$ which maps H_1 to H_2. The following question can now be asked.

> Suppose $n = 2k$ and that H_1 and H_2 are isomorphic hyperplanes of $A_{n-1,k}(\mathbb{F})$. Does there exist an isomorphism of $A_{n-1,k}(\mathbb{F})$ which is induced by a collineation of the ambient projective space which maps H_1 to H_2?

The answer to this question is affirmative for all the pairs $\{H_1, H_2\}$ of isomorphic hyperplanes of $A_{n-1,k}(\mathbb{F})$, $n = 2k$, which we will consider in Sections 6 and 7. The answer is however not always affirmative as the counter example in the following proposition shows.

Proposition 5.6 *Let V be an 8-dimensional vector space over a field $\mathbb{F}$ with ordered basis $B = (\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_8)$ and put $\alpha_1 := \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 \wedge \bar{e}_4 + \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_5 \wedge \bar{e}_6 + \bar{e}_3 \wedge \bar{e}_4 \wedge \bar{e}_5 \wedge \bar{e}_7$, $\alpha_2 := \bar{e}_5 \wedge \bar{e}_6 \wedge \bar{e}_7 \wedge \bar{e}_8 + \bar{e}_3 \wedge \bar{e}_4 \wedge \bar{e}_7 \wedge \bar{e}_8 - \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_6 \wedge \bar{e}_8$. Let H_i, $i \in \{1,2\}$, denote the hyperplane of $A_{7,4}(\mathbb{F})$ which has α_i as representative vector. Then:*

(1) H_1 and H_2 are isomorphic hyperplanes;

(2) there exists no automorphism of $A_{7,4}(\mathbb{F})$ induced by a collineation of $\mathrm{PG}(V)$ which maps H_1 to H_2.

Proof. We notice that α_2 is the dual vector of α_1 with respect to B. So, by Proposition 5.5, the hyperplanes H_1 and H_2 are isomorphic.

Notice that $\alpha_1 \wedge (a_1\bar{e}_1 + a_2\bar{e}_2 + \cdots + a_8\bar{e}_8) = 0$ if and only if $a_1 = a_2 = \cdots = a_8 = 0$ and $\alpha_2 \wedge (a_1\bar{e}_1 + a_2\bar{e}_2 + \cdots + a_8\bar{e}_8) = 0$ if and only if $a_1 = a_2 = \cdots = a_7 = 0$. So, by Proposition 5.3, α_1 and α_2 are not semi-equivalent. Notice also that if ψ is an automorphism of $\mathbb{F}$, then $\alpha_1^{\psi_B} = \alpha_1$ and $\alpha_2^{\psi_B} = \alpha_2$. Propositions 5.2 and 5.4 now imply that there exists no automorphism of $A_{7,4}(\mathbb{F})$ induced by a collineation of $\mathrm{PG}(V)$ which maps H_1 to H_2. ■

6 Hyperplanes arising from regular spreads of projective spaces

6.1 Regular spreads

Let $\mathrm{PG}(3,\mathbb{F})$ be a 3-dimensional projective space over a field $\mathbb{F}$. A *regulus* of $\mathrm{PG}(3,\mathbb{F})$ is a set $\mathcal{R}$ of mutually disjoint lines of $\mathrm{PG}(3,\mathbb{F})$ satisfying the following two properties:

- If a line L of $\mathrm{PG}(3,\mathbb{F})$ meets three distinct lines of $\mathcal{R}$, then L meets every line of $\mathcal{R}$;

- If a line L of $\mathrm{PG}(3,\mathbb{F})$ meets three distinct lines of $\mathcal{R}$, then every point of L is incident with (exactly) one line of $\mathcal{R}$.

Any three mutually disjoint lines L_1, L_2, L_3 of $\mathrm{PG}(3,\mathbb{F})$ are contained in a unique regulus which we will denote by $\mathcal{R}(L_1, L_2, L_3)$. The union of all lines of $\mathcal{R}(L_1, L_2, L_3)$ is a nonsingular quadric of Witt index 2 of $\mathrm{PG}(3,\mathbb{F})$.

Let $n \in \mathbb{N} \setminus \{0,1,2\}$ and $\mathbb{F}$ a field. A *spread* of the projective space $\mathrm{PG}(n,\mathbb{F})$ is a set of lines which determines a partition of the point set of $\mathrm{PG}(n,\mathbb{F})$. A spread S is called *regular* if the following two conditions are satisfied:

(R1) If π is a 3-dimensional subspace of $\mathrm{PG}(n,\mathbb{F})$ containing two elements of S, then the elements of S contained in π determine a spread of π;

(R2) If L_1, L_2 and L_3 are three distinct lines of S which are contained in some 3-dimensional subspace, then $\mathcal{R}(L_1, L_2, L_3) \subseteq S$.

6.2 Classification of regular spreads

Let $n \in \mathbb{N} \setminus \{0,1\}$ and let $\mathbb{F}, \mathbb{F}'$ be fields such that $\mathbb{F}'$ is a quadratic extension of $\mathbb{F}$. Let V' be an n-dimensional vector space over $\mathbb{F}'$ with basis $\{\bar{e}_1, \ldots, \bar{e}_n\}$. We denote by V the set of all $\mathbb{F}$-linear combinations of the elements of $\{\bar{e}_1, \ldots, \bar{e}_n\}$. Then V can be regarded as an n-dimensional vector space over $\mathbb{F}$. We denote the projective spaces associated with V and V' by $\mathrm{PG}(V)$ and $\mathrm{PG}(V')$, respectively. Since every 1-dimensional subspace of $\mathrm{PG}(V)$ is contained in a unique 1-dimensional subspace of $\mathrm{PG}(V')$, we can regard the points of $\mathrm{PG}(V)$ as points of $\mathrm{PG}(V')$. So, $\mathrm{PG}(V)$ can be regarded as a sub-(projective)-geometry of $\mathrm{PG}(V')$. Any subgeometry of $\mathrm{PG}(V')$ which can be obtained in this way is called a *Baer-$\mathbb{F}$-subgeometry* of $\mathrm{PG}(V')$. Notice also that every subspace π of $\mathrm{PG}(V)$ generates a subspace π' of $\mathrm{PG}(V')$ of the same dimension as π. Every point p of $\mathrm{PG}(V')$ not contained in $\mathrm{PG}(V)$ is contained in a unique line of $\mathrm{PG}(V')$ which intersects $\mathrm{PG}(V)$ in a line of $\mathrm{PG}(V)$, i.e. there exists a unique line L of $\mathrm{PG}(V)$ such that $p \in L'$. We call L the line of $\mathrm{PG}(V)$ *induced* by p.

Suppose $\mathbb{F}'$ is a separable (and hence also Galois) extension of $\mathbb{F}$ and let ψ denote the unique nontrivial element in $Gal(\mathbb{F}'/\mathbb{F})$. For every vector $\bar{x} = \sum_{i=1}^n k_i \bar{e}_i$ of V', we define $\bar{x}^\psi := \sum_{i=1}^n k_i^\psi \bar{e}_i$. For every point $p = \langle \bar{x} \rangle$ of $\mathrm{PG}(V')$, we define $p^\psi := \langle \bar{x}^\psi \rangle$ and for every subspace π of $\mathrm{PG}(V')$ we define $\pi^\psi := \{p^\psi \,|\, p \in \pi\}$. The subspace π^ψ is called *conjugate to* π with respect to ψ. Notice that if π is a subspace of $\mathrm{PG}(V)$, then $\pi'^\psi = \pi'$.

The following proposition is taken from Beutelspacher and Ueberberg [1, Theorem 1.2] and generalizes a result from Bruck [3].

Proposition 6.1 ([1]) *(a) Let $t \in \mathbb{N} \setminus \{0,1\}$ and let $\mathbb{F}, \mathbb{F}'$ be fields such that $\mathbb{F}'$ is a quadratic extension of $\mathbb{F}$. Regard $\mathrm{PG}(2t-1,\mathbb{F})$ as a Baer-$\mathbb{F}$-subgeometry of $\mathrm{PG}(2t-1,\mathbb{F}')$. Let π be a $(t-1)$-dimensional subspace of $\mathrm{PG}(2t-1,\mathbb{F}')$ disjoint from $\mathrm{PG}(2t-1,\mathbb{F})$. Then the set S_π of all lines of $\mathrm{PG}(2t-1,\mathbb{F})$ which are induced by the points of π is a regular spread of $\mathrm{PG}(2t-1,\mathbb{F})$.*

(b) *Suppose* $t \in \mathbb{N} \setminus \{0,1\}$ *and that* $\mathbb{F}$ *is a field. If* S *is a regular spread of the projective space* $\mathrm{PG}(2t-1,\mathbb{F})$*, then there exists a quadratic extension* $\mathbb{F}'$ *of* $\mathbb{F}$ *such that the following holds if we regard* $\mathrm{PG}(2t-1,\mathbb{F})$ *as a Baer-*$\mathbb{F}$*-subgeometry of* $\mathrm{PG}(2t-1,\mathbb{F}')$*:*

   (i) *If* $\mathbb{F}'$ *is a separable field extension of* $\mathbb{F}$*, then there are precisely two* $(t-1)$*-dimensional subspaces* π *of* $\mathrm{PG}(2t-1,\mathbb{F}')$ *disjoint from* $\mathrm{PG}(2t-1,\mathbb{F})$ *for which* $S = S_\pi$*.*

   (ii) *If* $\mathbb{F}'$ *is a non-separable field extension of* $\mathbb{F}$*, then there is exactly one* $(t-1)$*-dimensional subspace* π $\mathrm{PG}(2t-1,\mathbb{F}')$ *disjoint from* $\mathrm{PG}(2t-1,\mathbb{F})$ *for which* $S = S_\pi$*.*

Remark. In Proposition 6.1(b)(i), the two $(t-1)$-dimensional subspaces π_1, π_2 of $\mathrm{PG}(2t-1,\mathbb{F}')$ disjoint from $\mathrm{PG}(2t-1,\mathbb{F})$ for which $S = S_{\pi_1} = S_{\pi_2}$ are conjugate with respect to the unique nontrivial element ψ of $Gal(\mathbb{F}'/\mathbb{F})$. For, a line L of $\mathrm{PG}(2t-1,\mathbb{F})$ belongs to S_{π_1} if and only if L' intersects π_1, i.e., if and only if $L' = L'^{\psi}$ intersects π_1^{ψ}.

6.3 Some properties of regular spreads

Now, let $t \in \mathbb{N} \setminus \{0,1\}$, let $\mathbb{F}$ be a field and let $\overline{\mathbb{F}}$ be a given algebraic closure of $\mathbb{F}$. [In fact, the discussion below is also valid if we assume that $\overline{\mathbb{F}}$ is a splitting field of all quadratic polynomials over $\mathbb{F}$.] Let $\overline{V}$ be a $2t$-dimensional vector space over $\overline{\mathbb{F}}$ with basis $\{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{2t}\}$. For every subfield $\mathbb{F}'$ of $\overline{\mathbb{F}}$, let $V_{\mathbb{F}'}$ denote the set of all $\mathbb{F}'$-linear combinations of the elements of $\{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{2t}\}$. Then $V_{\mathbb{F}'}$ can be regarded as a $2t$-dimensional vector space over $\mathbb{F}'$. Clearly, we have $V_{\overline{\mathbb{F}}} = \overline{V}$. We denote the projective space $\mathrm{PG}(V_{\mathbb{F}'})$ associated to $V_{\mathbb{F}'}$ also by $\mathcal{P}_{\mathbb{F}'}$. Define $\overline{\mathcal{P}} := \mathcal{P}_{\overline{\mathbb{F}}}$, $\mathcal{P} := \mathcal{P}_{\mathbb{F}}$ and $V := V_{\mathbb{F}}$. Every 1-dimensional subspace of $V_{\mathbb{F}'}$ is contained in a unique 1-dimensional subspace of $\overline{V}$. This allows us to regard the points of $\mathcal{P}_{\mathbb{F}'}$ also as points of $\overline{\mathcal{P}}$. In this way, $\mathcal{P}_{\mathbb{F}'}$ is regarded as a sub-(projective)-geometry of $\overline{\mathcal{P}}$. Notice that if $\mathbb{F}'$ is a quadratic extension of $\mathbb{F}$, then $\mathcal{P}$ is a Baer-$\mathbb{F}$-subgeometry of $\mathcal{P}_{\mathbb{F}'}$. If α is a subspace of $\mathcal{P}$, then we denote by α' the subspace of $\overline{\mathcal{P}}$ (of the same dimension of α) generated by the points of α. The following is a rephrasing of Proposition 6.1(a).

Proposition 6.2 *Let* $\mathbb{F}'$ *be a quadratic extension of* $\mathbb{F}$ *contained in* $\overline{\mathbb{F}}$ *and let* π *be a* $(t-1)$*-dimensional subspace of* $\mathcal{P}_{\mathbb{F}'}$ *disjoint from* $\mathcal{P}$*. Then the set* S_π *of all lines of* $\mathcal{P}$ *which are induced by the points of* π *is a regular spread of* $\mathcal{P}$*.*

The following is a slight generalization of Proposition 6.1(b).

Proposition 6.3 *If* S *is a regular spread of* $\mathcal{P}$*, then there exists a unique quadratic extension* $\mathbb{F}'$ *of* $\mathbb{F}$ *contained in* $\overline{\mathbb{F}}$ *for which the projective space* $\mathcal{P}_{\mathbb{F}'}$ *has a* $(t-1)$*-dimensional subspace* π *disjoint from* $\mathcal{P}$ *such that* $S = S_\pi$*. If* $\mathbb{F}'$ *is a separable field extension of* $\mathbb{F}$*, then there are precisely two subspaces* π *for which this is the case. If* $\mathbb{F}'$ *is a non-separable field extension of* $\mathbb{F}$*, then there is precisely one subspace* π *for which this is the case.*

Proof. By Proposition 6.1(b), there exists some quadratic extension $\mathbb{F}_1'$ of $\mathbb{F}$ contained in $\overline{\mathbb{F}}$ and a subspace π_1 of $\mathcal{P}_{\mathbb{F}_1'}$ disjoint from $\mathcal{P}$ such that $S = S_{\pi_1}$. If $\mathbb{F}_1'$ is a non-separable extension of $\mathbb{F}$, then we define $\widetilde{\pi_1} := \pi_1$; otherwise, $\widetilde{\pi_1}$ denotes the $(t-1)$-dimensional subspace of $\mathcal{P}_{\mathbb{F}_1'}$ which is conjugate to π_1 with respect to the unique nontrivial element in $Gal(\mathbb{F}_1'/\mathbb{F})$.

Now, suppose that $\mathbb{F}_2'$ is some quadratic extension of $\mathbb{F}$ contained in $\overline{\mathbb{F}}$ and π_2 is some subspace of $\mathcal{P}_{\mathbb{F}_2'}$ disjoint from $\mathcal{P}$ such that $S = S_{\pi_2}$. We will prove that $\mathbb{F}_2' = \mathbb{F}_1'$ and that $\pi_2 \subseteq \pi_1 \cup \widetilde{\pi_1}$. The latter inclusion implies that $\pi_2 \in \{\pi_1, \widetilde{\pi_1}\}$ which is precisely what we need to prove.

Let p be an arbitrary point of π_2 and let L_1 denote the unique line of $\mathcal{P}$ for which $p \in L_1'$. There exist vectors $\bar{v}_1, \bar{w}_1 \in V$ such that $L_1' = \langle \bar{v}_1, \bar{w}_1 \rangle$ and $p = \langle \bar{v}_1 + \delta_2 \bar{w}_1 \rangle$ for some $\delta_2 \in \mathbb{F}_2' \setminus \mathbb{F}$ (recall $p \notin \mathcal{P}$ since $\pi_2 \cap \mathcal{P} = \emptyset$). Since $L_1 \in S = S_{\pi_1}$ and $\pi_1 \cap \mathcal{P} = \emptyset$, there exists a $\delta_1 \in \mathbb{F}_1' \setminus \mathbb{F}$ such that $\langle \bar{v}_1 + \delta_1 \bar{w}_1 \rangle \in \pi_1$. Let μ_1 and $\mu_2 \neq 0$ denote the unique elements of $\mathbb{F}$ such that $\delta_1^2 = \mu_1 \delta_1 + \mu_2$. Let L_2 denote an arbitrary line of $S \setminus \{L_1\}$. Since $S = S_{\pi_1}$, there exist vectors $\bar{v}_2, \bar{w}_2 \in V$ such that $\{\langle \bar{v}_2 + \delta_1 \bar{w}_2 \rangle\} = \pi_1 \cap L_2'$. Clearly, $L_2' = \langle \bar{v}_2, \bar{w}_2 \rangle$. Let L_3 denote the unique line of $S = S_{\pi_1}$ for which $L_3' \cap \pi_1 = \{\langle \bar{v}_1 + \bar{v}_2 + \delta_1(\bar{w}_1 + \bar{w}_2) \rangle\}$. Then $L_3' = \langle \bar{v}_1 + \bar{v}_2, \bar{w}_1 + \bar{w}_2 \rangle \subseteq \langle L_1', L_2' \rangle$. Let K denote the unique line through p meeting L_2' and L_3'. Then $K = \langle \bar{v}_1 + \delta_2 \bar{w}_1, \bar{v}_2 + \delta_2 \bar{w}_2 \rangle$. Since $\pi_2 \cap \mathcal{P} = \emptyset$, the subspace $\langle L_1', L_2' \rangle = \langle \bar{v}_1, \bar{v}_2, \bar{w}_1, \bar{w}_2 \rangle$ intersects π_2' in at most a line. Since $\{p\} = L_1' \cap \pi_2$, $L_2' \cap \pi_2$ and $L_3' \cap \pi_2$ are contained in $\pi_2' \cap \langle L_1', L_2' \rangle$, $\pi_2' \cap \langle L_1', L_2' \rangle$ is a line containing the points p, $L_2' \cap \pi_2$ and $L_3' \cap \pi_2$. So, $K = \pi_2' \cap \langle L_1', L_2' \rangle$. Now, consider the point $\langle (\bar{v}_1 + \delta_1 \bar{w}_1) + \delta_1(\bar{v}_2 + \delta_1 \bar{w}_2) \rangle = \langle (\bar{v}_1 + \mu_2 \bar{w}_2) + \delta_1(\bar{w}_1 + \bar{v}_2 + \mu_1 \bar{w}_2) \rangle$ of π_1. We see that $\langle \bar{v}_1 + \mu_2 \bar{w}_2, \bar{w}_1 + \bar{v}_2 + \mu_1 \bar{w}_2 \rangle \subseteq \langle L_1', L_2' \rangle$ is generated by some line of $S = S_{\pi_1}$. Since $S = S_{\pi_2}$ and $K = \pi_2' \cap \langle L_1', L_2' \rangle$, $\langle \bar{v}_1 + \mu_2 \bar{w}_2, \bar{w}_1 + \bar{v}_2 + \mu_1 \bar{w}_2 \rangle$ meets π_2 in a point of $K = \langle \bar{v}_1 + \delta_2 \bar{w}_1, \bar{v}_2 + \delta_2 \bar{w}_2 \rangle$. This implies that $\delta_2^2 = \mu_1 \delta_2 + \mu_2$. Hence, $\delta_2 \in \{\delta_1, \mu_1 - \delta_1\}$ and $\mathbb{F}_2' = \mathbb{F}(\delta_2) = \mathbb{F}(\delta_1) = \mathbb{F}_1'$. If $\mathbb{F}_1'$ is a non-separable field extension of $\mathbb{F}$, then $\delta_2 = \delta_1$ and hence $p \in \pi_1 = \pi_1 \cup \widetilde{\pi_1}$. If $\mathbb{F}_1'$ is a separable field extension, then $\delta_2 \in \{\delta_1, \delta_1^\psi\}$, where ψ denotes the unique nontrivial element in $Gal(\mathbb{F}_1'/\mathbb{F})$. If $\delta_2 = \delta_1$, then $p \in \pi_1$. If $\delta_2 = \delta_1^\psi$, then $p \in \widetilde{\pi} = \pi_1^\psi$. In any case, we have $p \in \pi_1 \cup \widetilde{\pi_1}$. ∎

Proposition 6.4 *Let $\mathbb{F}'$ be a quadratic extension of $\mathbb{F}$ contained in $\overline{\mathbb{F}}$ and let π_1, π_2 be two $(t-1)$-dimensional subspaces of $\mathcal{P}_{\mathbb{F}'}$ disjoint from $\mathcal{P}$. Then there exists a projectivity of $\mathcal{P}$ mapping S_{π_1} to S_{π_2}.*

Proof. Let δ be an arbitrary element of $\mathbb{F}' \setminus \mathbb{F}$. Then there exist unique $\mu_1 \in \mathbb{F}$ and $\mu_2 \in \mathbb{F} \setminus \{0\}$ such that $\delta^2 = \mu_1 \delta + \mu_2$. We can choose vectors $\bar{v}_1, \bar{w}_1, \ldots, \bar{v}_t, \bar{w}_t, \bar{v}_1', \bar{w}_1', \ldots, \bar{v}_t', \bar{w}_t'$ of V such that $\pi_1 = \langle \bar{v}_1 + \delta \bar{w}_1, \ldots, \bar{v}_t + \delta \bar{w}_t \rangle$ and $\pi_2 = \langle \bar{v}_1' + \delta \bar{w}_1', \ldots, \bar{v}_t' + \delta \bar{w}_t' \rangle$.

We prove that $\{\bar{v}_1, \bar{w}_1, \ldots, \bar{v}_t, \bar{w}_t\}$ is a basis of V. If this were not the case, then there exist $a_1, b_1, \ldots, a_t, b_t \in \mathbb{F}$ with $(a_1, b_1, \ldots, a_t, b_t) \neq (0, 0, \ldots, 0, 0)$ such that $a_1 \bar{v}_1 + b_1 \bar{w}_1 + \cdots + a_t \bar{v}_t + b_t \bar{w}_t = \bar{o}$. Now, put $k_i := a_i + \frac{b_i}{\mu_2} \delta$ for every $i \in \{1, \ldots, t\}$. Then $(k_1, \ldots, k_t) \neq (0, \ldots, 0)$ since $(a_1, b_1, \ldots, a_t, b_t) \neq (0, 0, \ldots, 0, 0)$. Since $k_1(\bar{v}_1 + \delta \bar{w}_1) + \cdots + k_t(\bar{v}_t + \delta \bar{w}_t) = \delta(a_1 \bar{w}_1 + \frac{b_1}{\mu_2} \bar{v}_1 + \frac{\mu_1}{\mu_2} b_1 \bar{w}_1 + \cdots + a_t \bar{w}_t + \frac{b_t}{\mu_2} \bar{v}_t + \frac{\mu_1}{\mu_2} b_t \bar{w}_t)$, the subspace π_1 is not disjoint from $\mathcal{P}$, a contradiction. So, $\{\bar{v}_1, \bar{w}_1, \ldots, \bar{v}_t, \bar{w}_t\}$ is a basis of V. In a similar way, one proves that $\{\bar{v}_1', \bar{w}_1', \ldots, \bar{v}_t', \bar{w}_t'\}$ is a basis of V.

Now, consider the unique element $\theta \in GL(V)$ mapping the ordered basis $(\bar{v}_1, \bar{w}_1, \dots, \bar{v}_t, \bar{w}_t)$ to $(\bar{v}'_1, \bar{w}'_1, \dots, \bar{v}'_t, \bar{w}'_t)$. Then θ extends to a unique element $\theta' \in GL(V_{\mathbb{F}'})$. The linear map θ' maps the subspace $\langle \bar{v}_1 + \delta\bar{w}_1, \dots, \bar{v}_t + \delta\bar{w}_t \rangle$ to the subspace $\langle \bar{v}'_1 + \delta\bar{w}'_1, \dots, \bar{v}'_t + \delta\bar{w}'_t \rangle$. So, the projectivity of $\mathrm{PG}(V)$ associated to θ maps S_{π_1} to S_{π_2}. ∎

Proposition 6.5 *Let $\mathbb{F}'_1$ and $\mathbb{F}'_2$ be two distinct quadratic extensions of $\mathbb{F}$ which are contained in $\overline{\mathbb{F}}$. Let π_i, $i \in \{1,2\}$, be a $(t-1)$-dimensional subspace of $\mathcal{P}_{\mathbb{F}'_i}$ disjoint from $\mathcal{P}$. Then the regular spreads S_{π_1} and S_{π_2} are not projectively equivalent.*

Proof. Suppose μ is a projectivity of $\mathcal{P}$ mapping S_{π_1} to S_{π_2}. Then μ can be extended to a projectivity μ_1 of $\mathcal{P}_{\mathbb{F}'_1}$. If $\pi_3 = \mu_1(\pi_1)$, then we necessarily have that $\mu(S_{\pi_1}) = S_{\pi_3}$. So, $S_{\pi_2} = S_{\pi_3}$. A contradiction is obtained from Proposition 6.3. ∎

Remark. Let ψ be an automorphism of $\mathbb{F}$ and let $a, b, c \in \mathbb{F}$ with $a \neq 0$. Then the quadratic polynomial $aX^2 + bX + c \in \mathbb{F}[X]$ is irreducible if and only if $a^\psi X^2 + b^\psi X + c^\psi \in \mathbb{F}[X]$ is irreducible. For, $\lambda \in \mathbb{F}$ is a root of $aX^2 + bX + c$ if and only if λ^ψ is a root of $a^\psi X^2 + b^\psi X + c^\psi$.

Lemma 6.6 *Let ψ be an automorphism of $\mathbb{F}$ and let $a_1X^2 + b_1X + c_1$ and $a_2X^2 + b_2X + c_2$ be two irreducible quadratic polynomials of $\mathbb{F}[X]$. Then the following are equivalent:*

(1) $a_1X^2 + b_1X + c_1$ and $a_2X^2 + b_2X + c_2$ define the same quadratic extension of $\mathbb{F}$ in $\overline{\mathbb{F}}$;

(2) $a_1^\psi X^2 + b_1^\psi X + c_1^\psi$ and $a_2^\psi X^2 + b_2^\psi X + c_2^\psi$ define the same quadratic extension of $\mathbb{F}$ in $\overline{\mathbb{F}}$.

Proof. By symmetry, we must only prove the implication (1) $\Rightarrow$ (2). We may suppose that $a_1 = a_2 = 1$. (Otherwise, divide the respective polynomials by their leading coefficients.)

Let $\delta_i \in \overline{\mathbb{F}}$, $i \in \{1,2\}$, be a root of the polynomial $X^2 + b_iX + c_i$. The quadratic polynomials $X^2 + b_1X + c_1$ and $X^2 + b_2X + c_2$ define the same quadratic extension of $\mathbb{F}$ (in $\overline{\mathbb{F}}$) if and only if there exist $\lambda, \mu \in \mathbb{F}$ with $\lambda \neq 0$ such that $\delta_2 = \lambda \cdot \delta_1 + \mu$. If this is the case, then the quadratic polynomials $X^2 + b_1X + c_1$ and $(\lambda X + \mu)^2 + b_2(\lambda X + \mu) + c_2$ are proportional. So, $X^2 + b_1X + c_1$ and $X^2 + b_2X + c_2$ define the same quadratic extension of $\mathbb{F}$ (in $\overline{\mathbb{F}}$) if and only if there exist $\lambda, \mu \in \mathbb{F}$ with $\lambda \neq 0$ such that $b_1 = \frac{2\mu + b_2}{\lambda}$ and $c_1 = \frac{\mu^2 + b_2\mu + c_2}{\lambda^2}$. So, if $X^2 + b_1X + c_1$ and $X^2 + b_2X + c_2$ define the same quadratic extension of $\mathbb{F}$ (in $\overline{\mathbb{F}}$), then there exist $\lambda, \mu \in \mathbb{F}$ with $\lambda \neq 0$ such that $b_1^\psi = \frac{2\mu^\psi + b_2^\psi}{\lambda^\psi}$ and $c_1^\psi = \frac{(\mu^\psi)^2 + b_2^\psi \mu^\psi + c_2^\psi}{(\lambda^\psi)^2}$. As explained above, this implies that also the polynomials $X^2 + b_1^\psi X + c_1^\psi$ and $X^2 + b_2^\psi X + c_2^\psi$ define the same quadratic extension of $\mathbb{F}$ (in $\overline{\mathbb{F}}$). ∎

Definition. Now, let $\mathcal{F}$ denote the set of all quadratic extensions of $\mathbb{F}$ which are contained in $\overline{\mathbb{F}}$. Define the following relation R on the set $\mathcal{F}$. If $F_1, F_2 \in \mathcal{F}$, then $(F_1, F_2) \in R$ if and only if there exist $b_1, c_1 \in \mathbb{F}$ and an automorphism ψ of $\mathbb{F}$ such that $\mathbb{F}_1 \subseteq \overline{\mathbb{F}}$ is the splitting field of $X^2 + b_1X + c_1$ and $\mathbb{F}_2 \subseteq \overline{\mathbb{F}}$ is the splitting field of $X^2 + b_1^\psi X + c_1^\psi$. Using Lemma 6.6, it is easily seen that R is an equivalence relation.

Proposition 6.7 *Let $\mathbb{F}'_1$ and $\mathbb{F}'_2$ be two distinct quadratic extensions of $\mathbb{F}$ which are contained in $\overline{\mathbb{F}}$. Let π_i, $i \in \{1,2\}$, be a $(t-1)$-dimensional subspace of $\mathcal{P}_{\mathbb{F}'_i}$ disjoint from $\mathcal{P}$. Then there exists a collineation of $\mathrm{PG}(V)$ mapping S_{π_1} to S_{π_2} if and only if $(\mathbb{F}'_1, \mathbb{F}'_2) \in R$.*

Proof. Let $\delta_1 \in \mathbb{F}'_1 \setminus \mathbb{F}$ and suppose $X^2 + b_1 X + c_1 \in \mathbb{F}[X]$ has δ_1 as root.

Suppose ψ is an automorphism of $\mathbb{F}$. Since $X^2 + b_1X + c_1$ is an irreducible polynomial of $\mathbb{F}[X]$, also the polynomial $X^2 + b_1^\psi X + c_1^\psi \in \mathbb{F}[X]$ is irreducible. Let $\mathbb{F}_2(\psi) \subseteq \overline{\mathbb{F}}$ denote the quadratic extension of $\mathbb{F}$ defined by $X^2 + b_1^\psi X + c_1^\psi$ and let $\delta_2 \in \mathbb{F}_2(\psi)$ be a root of $X^2 + b_1^\psi X + c_1^\psi$. The map $\overline{\psi} : \lambda_1 + \lambda_2\delta_1 \mapsto \lambda_1^\psi + \lambda_2^\psi \delta_2$ $(\lambda_1, \lambda_2 \in \mathbb{F})$ defines an isomorphism between the fields $\mathbb{F}'_1$ and $\mathbb{F}_2(\psi)$. So, the map $\sum_{i=1}^{2t} k_i \bar{e}_i \mapsto \sum_{i=1}^{2t} k_i^{\overline{\psi}} \bar{e}_i$ is a semi-linear map between the $\mathbb{F}'_1$-vector space $V_{\mathbb{F}'_1}$ and the $\mathbb{F}_2(\psi)$-vector space $V_{\mathbb{F}_2(\psi)}$. For every vector $\bar{x} = \sum_{i=1}^{2t} k_i \bar{e}_i$ of V, we define $\bar{x}^\psi = \sum_{i=1}^{2t} k_i^\psi \bar{e}_i$. If we put $\pi_1 = \langle \bar{v}_1 + \delta_1 \bar{w}_1, \bar{v}_2 + \delta_1 \bar{w}_2, \dots, \bar{v}_t + \delta_1 \bar{w}_t \rangle$ for some basis $\{\bar{v}_1, \bar{w}_1, \dots, \bar{v}_t, \bar{w}_t\}$ of V, then the following holds for linearly independent vectors $\bar{u}_1, \bar{u}_2$ of V. The line $\langle \bar{u}_1, \bar{u}_2 \rangle$ meets π_1 if and only if the line $\langle \bar{u}_1^\psi, \bar{u}_2^\psi \rangle$ meets $\langle \bar{v}_1^\psi + \delta_2 \bar{w}_1^\psi, \dots, \bar{v}_t^\psi + \delta_2 \bar{w}_t^\psi \rangle$. Clearly, $\langle \bar{v}_1^\psi + \delta_2 \bar{w}_1^\psi, \dots, \bar{v}_t^\psi + \delta_2 \bar{w}_t^\psi \rangle$ is a $(t-1)$-dimensional subspace of $\mathcal{P}_{\mathbb{F}_2(\psi)}$ disjoint from $\mathcal{P}$.

By Propositions 6.4 and 6.5, we can now conclude that there exists a collineation of $\mathrm{PG}(V)$ mapping S_{π_1} to S_{π_2} if and only if $\mathbb{F}'_2 = \mathbb{F}_2(\psi)$ for some automorphism ψ of $\mathbb{F}$, i.e. if and only if $(\mathbb{F}'_1, \mathbb{F}'_2) \in R$. ∎

6.4 Two lemmas

Let V be a vector space of dimension $n \geq 2$ over a field $\mathbb{F}$, let $k \in \{1, \dots, n-1\}$ and let $A_{n-1,k}(\mathbb{F})$ be the Grassmannian of the $(k-1)$-dimensional subspaces of $\mathrm{PG}(V)$.

Lemma 6.8 *Suppose X_1 and X_2 are two subspaces of $A_{n-1,k}(\mathbb{F})$ such that $X_1 \subsetneq X_2$ and there are no subspaces satisfying $X_1 \subsetneq X_3 \subsetneq X_2$. Let W_i, $i \in \{1,2\}$, denote the subspace of $\bigwedge^k V$ generated by all k-vectors $\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_k$, were $\langle \bar{v}_1, \bar{v}_2, \dots, \bar{v}_k \rangle$ is some point of X_i. Then W_1 has co-dimension at most 1 in W_2.*

Proof. Let $\langle \bar{v}_1, \dots, \bar{v}_k \rangle$ be an element of $X_2 \setminus X_1$. Then $\langle W_1, \bar{v}_1 \wedge \cdots \wedge \bar{v}_k \rangle \subseteq W_2$. Since the Grassmann embedding e_{gr} maps lines of $A_{n-1,k}(\mathbb{F})$ to lines of $\mathrm{PG}(\bigwedge^k V)$, the set of all points $\langle \bar{w}_1, \dots, \bar{w}_k \rangle$ of $A_{n-1,k}(\mathbb{F})$ satisfying $\bar{w}_1 \wedge \cdots \wedge \bar{w}_k \in \langle W_1, \bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_k \rangle$ is a subspace of $A_{n-1,k}(\mathbb{F})$ containing $X_1 \cup \{\langle \bar{v}_1, \dots, \bar{v}_k \rangle\}$ and hence also X_2. It follows that $W_2 \subseteq \langle W_1, \bar{v}_1 \wedge \cdots \wedge \bar{v}_k \rangle$. Hence, $W_2 = \langle W_1, \bar{v}_1 \wedge \cdots \wedge \bar{v}_k \rangle$ and W_1 has co-dimension at most 1 in W_2. ∎

Lemma 6.9 *Let α_1 and α_2 be two linearly independent $(n-k)$-vectors of V. Then the subspace W of $\bigwedge^k V$ generated by all k-vectors $\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_k$ satisfying $\alpha_1 \wedge \bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_k = \alpha_2 \wedge \bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_k = 0$ has dimension $\binom{n}{k} - 2$.*

Proof. Let W' denote the subspace of $\bigwedge^k V$ generated by all k-vectors β satisfying $\alpha_1 \wedge \beta = \alpha_2 \wedge \beta = 0$. Since α_1 and α_2 are linearly independent, W' has dimension $\binom{n}{k} - 2$. Clearly, $W \subseteq W'$. Put $\alpha_3 = \alpha_1 + \alpha_2$ and let H_i, $i \in \{1,2,3\}$, denote the hyperplane of $A_{n-1,k}(\mathbb{F})$ which has α_i as a representative vector. Then H_1, H_2, H_3 are mutually distinct and $H_1 \cap H_2 = H_1 \cap H_3 = H_2 \cap H_3$ consists of all $(k-1)$-dimensional subspaces $\langle \bar{v}_1, \ldots, \bar{v}_k \rangle$ of $\mathrm{PG}(V)$ satisfying $\bar{v}_1 \wedge \cdots \wedge \bar{v}_k \in W$. Since H_1 and H_2 are distinct maximal subspaces of $A_{n-1,k}(\mathbb{F})$, $H_1 \cap H_2$ is not a maximal subspace of $A_{n-1,k}(\mathbb{F})$. Since H_3 is a maximal subspace, $H_1 \cap H_2$ is properly contained in H_3. By De Bruyn [8, Lemma 2.2], $H_1 \cap H_2$ is a maximal proper subspace of H_3. Since H_3 is also a maximal proper subspace of $A_{n-1,k}(\mathbb{F})$, W has co-dimension at most 2 in $\bigwedge^k V$ by Lemma 6.8. Since $W \subseteq W'$ and $\dim(W') = \binom{n}{k} - 2$, we necessarily have $W = W'$ and $\dim(W) = \binom{n}{k} - 2$. ■

6.5 Hyperplanes from regular spreads

Let V be an n-dimensional vector space over a field $\mathbb{F}$ and suppose $n = 2m \geq 4$ is even. Let $A_{n-1,m}(\mathbb{F})$ denote the Grassmannian of the $(m-1)$-dimensional subspaces of $\mathrm{PG}(V)$. For every spread S of $\mathrm{PG}(V)$, let X_S denote the set of all $(m-1)$-dimensional subspaces of $\mathrm{PG}(V)$ which contain at least one line of S, and let $\mathcal{H}_S$ denote the set of hyperplanes of $A_{n-1,m}(\mathbb{F})$ containing X_S. A hyperplane of $A_{n-1,m}(\mathbb{F})$ is said to be of *spread-type* if it contains some set X_S where S is a regular spread of $\mathrm{PG}(V)$.

Proposition 6.10 *The following holds for a regular spread S of* $\mathrm{PG}(V)$*.*

(1) *$\mathcal{H}_S \neq \emptyset$ and the representative vectors of the elements of $\mathcal{H}_S$ are precisely the nonzero vectors of a certain 2-dimensional subspace of $\bigwedge^m V$.*

(2) *If $H \in \mathcal{H}_S$, then every line of $A_{n-1,m}(\mathbb{F})$ contained in H intersects X_S in either a singleton or the whole line.*

(3) *If H_1 and H_2 are two distinct hyperplanes of $\mathcal{H}_S$, then $H_1 \cap H_2 = X_S$.*

(4) *If H_1 and H_2 are two distinct hyperplanes of $\mathcal{H}_S$, then there exists an automorphism of $A_{n-1,m}(\mathbb{F})$ induced by a projectivity of* $\mathrm{PG}(V)$ *mapping H_1 to H_2.*

Proof. Suppose that $\mathbb{F}'$ is a quadratic extension of $\mathbb{F}$, that V' is an n-dimensional vector space over $\mathbb{F}'$ with basis $\{\bar{e}_1, \ldots, \bar{e}_n\}$, that V is the set of all $\mathbb{F}$-linear combinations of the elements of $\{\bar{e}_1, \ldots, \bar{e}_n\}$ and that π is an $(m-1)$-dimensional subspace of $\mathrm{PG}(V')$ disjoint from $\mathrm{PG}(V)$ such that S consists of all lines of $\mathrm{PG}(V)$ which are induced by the points of π. Let δ be an arbitrary element of $\mathbb{F}' \setminus \mathbb{F}$ and let μ_1, μ_2 be the unique elements of $\mathbb{F}$ such that $\delta^2 = \mu_1 \delta + \mu_2$. Then $\mu_2 \neq 0$. There exist vectors $\bar{v}_1, \bar{w}_1, \ldots, \bar{v}_m, \bar{w}_m$ of V such that $\pi = \langle \bar{v}_1 + \delta \bar{w}_1, \bar{v}_2 + \delta \bar{w}_2, \ldots, \bar{v}_m + \delta \bar{w}_m \rangle$. We know, see the proof of Proposition 6.4, that $\{\bar{v}_1, \bar{w}_1, \ldots, \bar{v}_m, \bar{w}_m\}$ is a basis of V. Put $\alpha = (\bar{v}_1 + \delta \bar{w}_1) \wedge (\bar{v}_2 + \delta \bar{w}_2) \wedge \cdots \wedge (\bar{v}_m + \delta \bar{w}_m) = \alpha^{(1)} + \delta \alpha^{(2)}$, where $\alpha^{(1)}, \alpha^{(2)} \in \bigwedge^m V$. The vectors $\alpha^{(1)}$ and $\alpha^{(2)}$ are linearly independent: $\alpha^{(1)}$ contains a term in $\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_m$, while $\alpha^{(2)}$ does not contain such a term; $\alpha^{(2)}$ contains a term in $\bar{w}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 \wedge \cdots \wedge \bar{v}_m$, while $\alpha^{(1)}$ does not contain such a term.

Let τ be an $(m-1)$-dimensional subspace of $\mathrm{PG}(V)$ and let τ' be the $(m-1)$-dimensional subspace of $\mathrm{PG}(V')$ generated by the points of τ. If $\tau \in X_S$, then τ' meets π. Conversely, suppose that τ' meets π and let p be an arbitrary point in $\tau' \cap \pi$. Then there exists a unique line L'_p of τ' through p which meets τ in a line L_p of τ. Clearly, $L_p \in S$ and hence $\tau \in X_S$.

So, the set X_S consists of all $(m-1)$-dimensional subspaces $\tau = \langle \bar{u}_1, \ldots, \bar{u}_m \rangle$ of $\mathrm{PG}(V)$ for which τ' meets π, i.e. which satisfy $\alpha \wedge \bar{u}_1 \wedge \bar{u}_2 \wedge \cdots \wedge \bar{u}_m = 0$. Hence, $\langle \bar{u}_1, \ldots, \bar{u}_m \rangle \in X_S$

if and only if $\alpha^{(1)} \wedge \bar{u}_1 \wedge \bar{u}_2 \wedge \cdots \wedge \bar{u}_m = \alpha^{(2)} \wedge \bar{u}_1 \wedge \bar{u}_2 \wedge \cdots \wedge \bar{u}_m = 0$. By Lemma 6.9, the subspace W_S of $\bigwedge^m V$ generated by all m-vectors of the form $\bar{u}_1 \wedge \bar{u}_2 \wedge \cdots \wedge \bar{u}_m$, where $\langle \bar{u}_1, \bar{u}_2, \ldots, \bar{u}_m \rangle \in X_S$ has co-dimension 2 in $\bigwedge^m V$. This subspace is generated by all m-vectors $\bar{u}_1 \wedge \bar{u}_2 \wedge \cdots \wedge \bar{u}_m$ of V which satisfy $\alpha^{(1)} \wedge \bar{u}_1 \wedge \bar{u}_2 \wedge \cdots \wedge \bar{u}_m = \alpha^{(2)} \wedge \bar{u}_1 \wedge \bar{u}_2 \wedge \cdots \wedge \bar{u}_m = 0$. So, the hyperplanes of $\mathcal{H}_S$ are precisely those hyperplanes of $A_{n-1,m}(\mathbb{F})$ who have a representative vector of the form $\lambda_1 \alpha^{(1)} + \lambda_2 \alpha^{(2)}$, where $(\lambda_1, \lambda_2) \in \mathbb{F}^2 \setminus \{(0,0)\}$. This proves Claim (1).

If H is a hyperplane of $\mathcal{H}_S$, then by Proposition 2.4(2), there exists a hyperplane W_H of $\bigwedge^k V$ such that a point $\langle \bar{u}_1, \bar{u}_2, \ldots, \bar{u}_m \rangle$ of $A_{n-1,m}(\mathbb{F})$ belongs to H if and only if $\bar{u}_1 \wedge \bar{u}_2 \wedge \cdots \wedge \bar{u}_m \in W_H$. Clearly, W_H contains W_S as a hyperplane. Since e_{gr} maps lines of $A_{n-1,m}(\mathbb{F})$ to lines of $\mathrm{PG}(\bigwedge^k V)$, every line of $A_{n-1,m}(\mathbb{F})$ contained in H intersects X_S in either a singleton or the whole line. This proves Claim (2).

Suppose H_1 and H_2 are two distinct elements of $\mathcal{H}_S$. Let $\lambda_1, \lambda_2, \lambda_1', \lambda_2' \in \mathbb{F}$ such that $\lambda_1 \alpha^{(1)} + \lambda_2 \alpha^{(2)}$ is a representative vector of H_1 and $\lambda_1' \alpha^{(1)} + \lambda_2' \alpha^{(2)}$ is a representative vector of H_2. Suppose $H_1 \neq H_2$. Then (λ_1, λ_2) and (λ_1', λ_2') are linearly independent elements of $\mathbb{F}^2$. The set $H_1 \cap H_2$ consists of all $(m-1)$-dimensional subspaces $\langle \bar{u}_1, \bar{u}_2, \ldots, \bar{u}_m \rangle$ of $\mathrm{PG}(W)$ which satisfy $(\lambda_1 \alpha^{(1)} + \lambda_2 \alpha^{(2)}) \wedge \bar{u}_1 \wedge \bar{u}_2 \wedge \cdots \wedge \bar{u}_m = (\lambda_1' \alpha^{(1)} + \lambda_2' \alpha^{(2)}) \wedge \bar{u}_1 \wedge \bar{u}_2 \wedge \cdots \wedge \bar{u}_m = 0$, or equivalently, $\alpha^{(1)} \wedge \bar{u}_1 \wedge \bar{u}_2 \wedge \cdots \wedge \bar{u}_m = \alpha^{(2)} \wedge \bar{u}_1 \wedge \bar{u}_2 \wedge \cdots \wedge \bar{u}_m = 0$. Hence, $H_1 \cap H_2 = X_S$. This proves Claim (3).

Put $(\bar{v}_2 + \delta \bar{w}_2) \wedge \cdots \wedge (\bar{v}_m + \delta \bar{w}_m) = \beta^{(1)} + \delta \beta^{(2)}$. Then $\alpha = (\bar{v}_1 + \delta \bar{w}_1) \wedge (\beta^{(1)} + \delta \beta^{(2)}) = \bar{v}_1 \wedge \beta^{(1)} + \mu_2 \cdot \bar{w}_1 \wedge \beta^{(2)} + \delta \cdot (\bar{w}_1 \wedge \beta^{(1)} + \bar{v}_1 \wedge \beta^{(2)} + \mu_1 \cdot \bar{w}_1 \wedge \beta^{(2)})$. So,

$$\begin{aligned} \alpha^{(1)} &= \bar{v}_1 \wedge \beta^{(1)} + \mu_2 \cdot \bar{w}_1 \wedge \beta^{(2)}, \\ \alpha^{(2)} &= \bar{w}_1 \wedge \beta^{(1)} + \bar{v}_1 \wedge \beta^{(2)} + \mu_1 \cdot \bar{w}_1 \wedge \beta^{(2)}. \end{aligned}$$

Now, let $a, b \in \mathbb{F}$ with $(a,b) \neq (0,0)$. Since the polynomials $X^2 - \mu_1 X - \mu_2$ and $X^2 + \mu_1 X - \mu_2$ are irreducible in $\mathbb{F}[X]$ (recall $\delta^2 = \mu_1 \delta + \mu_2$ with $\delta \in \mathbb{F}' \setminus \mathbb{F}$),

$$\begin{vmatrix} a & b\mu_2 \\ b & a + b\mu_1 \end{vmatrix} = a^2 + ab\mu_1 - b^2 \mu_2 \neq 0.$$

So, the linear map θ defined by

$$\begin{aligned} \theta(\bar{v}_1) &= a \cdot \bar{v}_1 + b\mu_2 \cdot \bar{w}_1, \\ \theta(\bar{w}_1) &= b \cdot \bar{v}_1 + (a + b\mu_1) \bar{w}_1, \\ \theta(\bar{v}_j) &= \bar{v}_j, \ j \in \{2, \ldots, m\}, \\ \theta(\bar{w}_j) &= \bar{w}_j, \ j \in \{2, \ldots, m\}. \end{aligned}$$

belongs to $GL(V)$. We have

$$\begin{aligned} \wedge^m(\theta)(\alpha^{(1)}) &= (a \cdot \bar{v}_1 + b\mu_2 \cdot \bar{w}_1) \wedge \beta^{(1)} + \mu_2 \Big(b \cdot \bar{v}_1 + (a + b\mu_1) \bar{w}_1 \Big) \wedge \beta^{(2)} \\ &= a \cdot \alpha^{(1)} + b\mu_2 \cdot \alpha^{(2)}. \end{aligned}$$

So, all m-vectors $\lambda_1 \alpha^{(1)} + \lambda_2 \alpha^{(2)}$, $(\lambda_1, \lambda_2) \in \mathbb{F}^2 \setminus \{(0,0)\}$, are equivalent. Claim (4) then follows from Proposition 5.2. ∎

Proposition 6.11 *Let S be a regular spread of* $\mathrm{PG}(V)$ *and let H be a hyperplane of* $A_{n-1,m}(\mathbb{F})$ *containing X_S. If L is a line of* $\mathrm{PG}(V)$ *not contained in S, then there exists an $(m-1)$-dimensional subspace through L not belonging to H.*

Proof. Obviously, the proposition holds if $m = 2$. So, we will suppose that $m \geq 3$. Suppose $\mathbb{F}'$ is a quadratic extension of $\mathbb{F}$, that V' is an n-dimensional vector space over $\mathbb{F}'$ with basis $\{\bar{e}_1, \ldots, \bar{e}_n\}$, that V is the set of all $\mathbb{F}$-linear combinations of the elements of $\{\bar{e}_1, \ldots, \bar{e}_n\}$ and that π is an $(m-1)$-dimensional subspace of $\mathrm{PG}(V')$ disjoint from $\mathrm{PG}(V)$ such that S consists of all lines of $\mathrm{PG}(V)$ which are induced by the points of π. Let δ be an arbitrary element of $\mathbb{F}' \setminus \mathbb{F}$ and suppose $L = p_1p_2$ for certain distinct points p_1 and p_2 of $\mathrm{PG}(V)$. Let L_i, $i \in \{1,2\}$, denote the unique line of S through p_i. Then there exist vectors $\bar{w}_1, \bar{w}_2 \in V$ such that $L_1' = \langle \bar{v}_1, \bar{w}_1 \rangle$, $L_2' = \langle \bar{v}_2, \bar{w}_2 \rangle$, $L_1' \cap \pi = \{\langle \bar{v}_1 + \delta\bar{w}_1 \rangle\}$ and $L_2' \cap \pi = \{\langle \bar{v}_2 + \delta\bar{w}_2 \rangle\}$. Put $p_1 = \langle k_1\bar{v}_1 + l_1\bar{w}_1 \rangle$ and $p_2 = \langle k_2\bar{v}_2 + l_2\bar{w}_2 \rangle$ where $(k_1, l_1), (k_2, l_2) \in \mathbb{F}^2 \setminus \{(0,0)\}$. Let $\bar{v}_3, \bar{w}_3, \ldots, \bar{v}_m, \bar{w}_m$ be vectors of V such that $\pi = \langle \bar{v}_1 + \delta\bar{w}_1, \bar{v}_2 + \delta\bar{w}_2, \ldots, \bar{v}_m + \delta\bar{w}_m \rangle$. Let π_1 be the $(m-2)$-dimensional subspace $\langle k_1\bar{v}_1 + l_1\bar{w}_1, k_2\bar{v}_2 + l_2\bar{w}_2, \bar{v}_3, \ldots, \bar{v}_{m-1} \rangle$ of $\mathrm{PG}(V)$ and let π_2 be the m-dimensional subspace $\langle k_1\bar{v}_1 + l_1\bar{w}_1, k_2\bar{v}_2 + l_2\bar{w}_2, \bar{v}_3, \ldots, \bar{v}_{m-1}, \bar{v}_m, \bar{w}_m \rangle$ of $\mathrm{PG}(V)$. Then $L(\pi_1, \pi_2)$ is a line of $A_{n-1,m}(\mathbb{F})$. If $L(\pi_1, \pi_2) \subseteq H$, then by Proposition 6.10(2), there exists some element $\pi_3 \in L(\pi_1, \pi_2)$ which belongs to X_S. So, there exists some $k \in \mathbb{F}$ such that $\pi = \langle \bar{v}_1 + \delta\bar{w}_1, \bar{v}_2 + \delta\bar{w}_2, \ldots, \bar{v}_m + \delta\bar{w}_m \rangle$ and $\langle k_1\bar{v}_1 + l_1\bar{w}_1, k_2\bar{v}_2 + l_2\bar{w}_2, \bar{v}_3, \ldots, \bar{v}_{m-1}, \bar{v}_m + k\bar{w}_m \rangle$ meet. But this is impossible since $\delta \not\in \mathbb{F}$. So, there exists some element of $L(\pi_1, \pi_2)$ not contained in H. Hence, there exists some $(m-1)$-dimensional subspace through L not belonging to H. ■

Proposition 6.12 *Let S_1 and S_2 be two regular spreads of* $\mathrm{PG}(V)$ *and let H_i, $i \in \{1,2\}$, be a hyperplane of* $A_{n-1,m}(\mathbb{F})$ *containing X_{S_i}. Then there exists an automorphism of* $A_{n-1,m}(\mathbb{F})$ *induced by a collineation (projectivity) of* $\mathrm{PG}(V)$ *mapping H_1 to H_2 if and only if there exists a collineation (projectivity) of* $\mathrm{PG}(V)$ *mapping S_1 to S_2.*

Proof. Suppose there exists a collineation (projectivity) η of $\mathrm{PG}(V)$ mapping S_1 to S_2. Then η induces an automorphism of $A_{n-1,m}(\mathbb{F})$ which maps H_1 to some hyperplane H_2' which contains X_{S_2}. Combining this with Proposition 6.10(4), we see that there exists an automorphism of $A_{n-1,m}(\mathbb{F})$ induced by a collineation (projectivity) of $\mathrm{PG}(V)$ which maps H_1 to H_2.

Conversely, suppose that there exists an automorphism of $A_{n-1,m}(\mathbb{F})$ induced by a collineation (projectivity) η of $\mathrm{PG}(V)$ which maps H_1 to H_2. Then H_2 contains $X_{S_1^\eta}$. Hence, $S_1^\eta = S_2$ by Proposition 6.11. ■

Lemma 6.13 (1) *Let $\mathbb{F}'$ be a quadratic extension of $\mathbb{F}$ and let $\delta \in \mathbb{F}' \setminus \mathbb{F}$. Let V' be an n-dimensional vector space over $\mathbb{F}'$ with ordered basis $B = (\bar{e}_1^+, \bar{e}_2^+, \ldots, \bar{e}_m^+, \bar{e}_1^-, \bar{e}_2^-, \ldots, \bar{e}_m^-)$. Let V denote the $\mathbb{F}$-vector space whose elements consist of all $\mathbb{F}$-linear combinations of the elements of $\{\bar{e}_1^+, \bar{e}_2^+, \ldots, \bar{e}_m^+, \bar{e}_1^-, \bar{e}_2^-, \ldots, \bar{e}_m^-\}$. Put $(\bar{e}_1^+ + \delta\bar{e}_1^-) \wedge (\bar{e}_2^+ + \delta\bar{e}_2^-) \wedge \cdots \wedge (\bar{e}_m^+ + \delta\bar{e}_m^-) = \alpha^{(1)} + \delta\alpha^{(2)}$ and $(\bar{e}_1^- - \delta\bar{e}_1^+) \wedge \cdots \wedge (\bar{e}_m^- - \delta\bar{e}_m^+) = \beta^{(1)} + \delta\beta^{(2)}$, where $\alpha^{(1)}, \alpha^{(2)}, \beta^{(1)}, \beta^{(2)} \in \bigwedge^m V$. Then the dual vectors of $\alpha^{(1)}$ and $\alpha^{(2)}$ with respect to B are respectively equal to $\beta^{(1)}$ and $\beta^{(2)}$.*

(2) *If θ is the element of $GL(V)$ defined by $\bar{e}_i^+ \mapsto \bar{e}_i^-, \bar{e}_i^- \mapsto -\bar{e}_i^+$, $\forall i \in \{1,\ldots,m\}$, then $\bigwedge^m(\theta)(\lambda_1 \cdot \alpha^{(1)} + \lambda_2 \cdot \alpha^{(2)}) = \lambda_1\beta^{(1)} + \lambda_2\beta^{(2)}$ for all $(\lambda_1,\lambda_2) \in \mathbb{F}^2$.*

Proof. (1) It suffices to prove that the dual vector of $\alpha^{(1)} + \delta\alpha^{(2)}$ with respect to B coincides with $\beta^{(1)} + \delta\beta^{(2)}$. The vector $\alpha^{(1)} + \delta\alpha^{(2)}$ can be written as the sum of 2^m terms. Each such term has the form $(\delta^{k_1}\bar{e}_1^{\epsilon_1}) \wedge (\delta^{k_2}\bar{e}_2^{\epsilon_2}) \wedge \cdots \wedge (\delta^{k_m}\bar{e}_m^{\epsilon_m})$, where $(k_i,\epsilon_i) \in \{(0,+),(1,-)\}$ for every $i \in \{1,\ldots,m\}$. By Proposition 3.2(2), the dual vector of this m-vector with respect to B is equal to $(-1)^N(\delta^{k_1}\bar{e}_1^{-\epsilon_1}) \wedge (\delta^{k_2}\bar{e}_2^{-\epsilon_2}) \wedge \cdots \wedge (\delta^{k_m}\bar{e}_m^{-\epsilon_m})$, where N is the total number of $i \in \{1,\ldots,m\}$ for which $\epsilon_i = -1$. The map $(\delta^{k_1}\bar{e}_1^{\epsilon_1}) \wedge \cdots \wedge (\delta^{k_m}\bar{e}_m^{\epsilon_m}) \mapsto (-1)^N(\delta^{k_1}\bar{e}_1^{-\epsilon_1}) \wedge \cdots \wedge (\delta^{k_m}\bar{e}_m^{-\epsilon_m})$ establishes a bijective correspondence between the set of 2^m terms occurring in $\alpha^{(1)} + \delta\alpha^{(2)}$ and the set of 2^m terms occurring in $\beta^{(1)} + \delta\beta^{(2)}$. Hence, $\beta^{(1)} + \delta\beta^{(2)}$ is the dual vector of $\alpha^{(1)} + \delta\alpha^{(2)}$ with respect to B, as we needed to prove.

(2) Clearly, we have that $\bigwedge^m(\theta)(\alpha^{(1)} + \delta\alpha^{(2)}) = \beta^{(1)} + \delta\beta^{(2)}$. So, $\bigwedge^m(\theta)(\alpha^{(1)}) = \beta^{(1)}$, $\bigwedge^m(\theta)(\alpha^{(2)}) = \beta^{(2)}$ and $\bigwedge^m(\theta)(\lambda_1 \cdot \alpha^{(1)} + \lambda_2 \cdot \alpha^{(2)}) = \lambda_1 \cdot \beta^{(1)} + \lambda_2 \cdot \beta^{(2)}$ for all $(\lambda_1,\lambda_2) \in \mathbb{F}^2$. ■

Let $\overline{\mathbb{F}}$ be a given algebraic closure of $\mathbb{F}$. For every quadratic extension $\mathbb{F}'$ of $\mathbb{F}$ contained in $\overline{\mathbb{F}}$, let $\mathcal{P}_{\mathbb{F}'}$ be a projective space as defined in Section 6.3.

Proposition 6.14 *Let H be a hyperplane of spread-type of $A_{n-1,m}(\mathbb{F})$. Then there exists an automorphism of $A_{n-1,m}(\mathbb{F})$ induced by a duality of $\mathrm{PG}(V)$ which maps H to itself.*

Proof. By Proposition 6.11, there exists a unique regular spread S_1 of $\mathrm{PG}(V)$ such that $X_{S_1} \subseteq H$. Let $B = (\bar{e}_1^+, \bar{e}_2^+, \ldots, \bar{e}_m^+, \bar{e}_1^-, \bar{e}_2^-, \ldots, \bar{e}_m^-)$ be an ordered basis of V, $\mathbb{F}'$ a quadratic extension of $\mathbb{F}$ and $\delta \in \mathbb{F}' \setminus \mathbb{F}$ such that the lines of S_1 are induced by the points of the subspace $\pi_1 = \langle \bar{e}_1^+ + \delta\bar{e}_1^-, \bar{e}_2^+ + \delta\bar{e}_2^-, \ldots, \bar{e}_m^+ + \delta\bar{e}_m^- \rangle$ of $\mathcal{P}_{\mathbb{F}'}$. Let π_2 be the subspace $\langle \bar{e}_1^- - \delta\bar{e}_1^+, \ldots, \bar{e}_m - \delta\bar{e}_m^+ \rangle$ of $\mathcal{P}_{\mathbb{F}'}$ and let S_2 be the regular spread of $\mathrm{PG}(V)$ whose lines are induced by the points of π_2. Then by Proposition 5.5, the proof of Proposition 6.10(1) and Lemma 6.13(1), there exists a polarity ν_B of $\mathrm{PG}(V)$ which maps $\mathcal{H}_{S_1}$ to $\mathcal{H}_{S_2}$. By Proposition 6.4, there exists a projectivity of $\mathrm{PG}(V)$ which maps S_1 to S_2. Hence, by Proposition 6.12, there exists an automorphism of $A_{n-1,m}(\mathbb{F})$ induced by a projectivity ν' of $\mathrm{PG}(V)$ which maps H^{ν_B} to H. Now, the duality $\nu' \circ \nu_B$ of $\mathrm{PG}(V)$ induces an automorphism of $A_{n-1,m}(\mathbb{F})$ which maps H to itself. ■

Corollary 6.15 (1) *If H_1 and H_2 are two hyperplanes of spread-type of $A_{n-1,m}(\mathbb{F})$, then H_1 and H_2 are isomorphic if and only if there exists an automorphism of $A_{n-1,m}(\mathbb{F})$ induced by a collineation of $\mathrm{PG}(V)$ which maps H_1 to H_2.*

(2) *Let S_1 and S_2 be two regular spreads of $\mathrm{PG}(V)$ and let $H_i \in \mathcal{H}_{S_i}$, $i \in \{1,2\}$. Then H_1 and H_2 are isomorphic if and only if there exists a collineation of $\mathrm{PG}(V)$ mapping S_1 to S_2.*

(3) *Let $\mathbb{F}'_i$, $i \in \{1,2\}$, be a quadratic extension of $\mathbb{F}$ contained in $\overline{\mathbb{F}}$, let π_i be an $(m-1)$-dimensional subspace of $\mathcal{P}_{\mathbb{F}'_i}$ disjoint from $\mathrm{PG}(V)$, let S_i be the regular spread of $\mathrm{PG}(V)$ whose lines are induced by the points of π_i and let $H_i \in \mathcal{H}_{S_i}$. Then H_1 and H_2*

are isomorphic if and only if $(\mathbb{F}'_1, \mathbb{F}'_2) \in R$*, where* R *is the equivalence relation as defined in Section 6.3.*

Proof. Claim (1) is a corollary of Proposition 5.1(2) and Proposition 6.14. Claim (2) is a corollary of Claim (1) and Proposition 6.12. Claim (3) is a corollary of Claim (2) and Proposition 6.7. ∎

6.6 Hyperplanes of spread-type of $A_{5,3}(\mathbb{F})$

Let V be a 6-dimensional vector space over a field $\mathbb{F}$ and let $A_{5,3}(\mathbb{F})$ denote the Grassmannian of the planes of $\mathrm{PG}(V)$. Let $\{\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4, \bar{e}_5, \bar{e}_6\}$ be a basis of V. Put $\mathcal{P} = \mathrm{PG}(V)$. Let $\overline{\mathbb{F}}$ be a given algebraic closure of $\mathbb{F}$.

Now, let $\mathbb{F}'$ be a given quadratic extension of $\mathbb{F}$ contained in $\overline{\mathbb{F}}$. Similarly, as in Section 6.3, we can construct a vector space $V_{\mathbb{F}'}$ over $\mathbb{F}'$. Let δ be an arbitrary element of $\mathbb{F}' \setminus \mathbb{F}$. Then δ is a root of a unique irreducible monic quadratic polynomial $q(X) = X^2 - aX - b \in \mathbb{F}[X]$. Since $b = -q(0) \neq 0 \neq q(1) = 1 - a - b$, the values $\mu_1 = a + b - 1$ and $\mu_2 = \frac{1-a-b}{b}$ are nonzero. The field $\mathbb{F}'$ is the splitting field (in $\overline{\mathbb{F}}$) of the quadratic polynomial $\mu_2 X^2 - (\mu_1\mu_2 + \mu_1 + \mu_2)X + \mu_1 = \mu_2(X^2 - aX - b) \in \mathbb{F}[X]$. We define

$$\alpha_{\mathbb{F}'} \;:=\; \mu_1 \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 + \mu_2 \cdot \bar{e}_4 \wedge \bar{e}_5 \wedge \bar{e}_6 + (\bar{e}_1 + \bar{e}_4) \wedge (\bar{e}_2 + \bar{e}_5) \wedge (\bar{e}_3 + \bar{e}_6).$$

Let $\pi_{\mathbb{F}'}$ be a plane of $\mathrm{PG}(V_{\mathbb{F}'})$ which is disjoint from $\mathcal{P}$ and let $S_{\mathbb{F}'}$ denote the regular spread of $\mathcal{P}$ whose lines are induced by the points of $\pi_{\mathbb{F}'}$. Let $H_{\mathbb{F}'}$ be a hyperplane of $A_{5,3}(\mathbb{F})$ containing all planes through a line of $S_{\mathbb{F}'}$.

Proposition 6.16 (1) *Any representative vector of* $H_{\mathbb{F}'}$ *is (semi-)equivalent with* $\alpha_{\mathbb{F}'}$*.*

(2) *If* $\mathbb{F}'_1$ *and* $\mathbb{F}'_2$ *are two distinct quadratic extensions of* $\mathbb{F}$ *which are contained in* $\overline{\mathbb{F}}$*, then* $\alpha_{\mathbb{F}'_1}$ *and* $\alpha_{\mathbb{F}'_2}$ *are not semi-equivalent.*

Proof. (1) Notice first that if $\lambda \in \mathbb{F} \setminus \{0\}$, then $\lambda \cdot \alpha_{\mathbb{F}'}$ is equivalent with $\alpha_{\mathbb{F}'}$. For $\bigwedge^3(\theta)(\alpha_{\mathbb{F}'}) = \lambda \cdot \alpha_{\mathbb{F}'}$, where θ denotes the following map of $GL(V)$: $\bar{e}_1 \mapsto \lambda \cdot \bar{e}_1$, $\bar{e}_2 \mapsto \bar{e}_2$, $\bar{e}_3 \mapsto \bar{e}_3$, $\bar{e}_4 \mapsto \lambda \cdot \bar{e}_4$, $\bar{e}_5 \mapsto \bar{e}_5$, $\bar{e}_6 \mapsto \bar{e}_6$. So, it suffices to prove that any representative vector of $H_{\mathbb{F}'}$ is semi-equivalent with $\alpha_{\mathbb{F}'}$.

Notice that $\delta^2 = a\delta + b$ and $\delta^3 = (a^2+b)\delta + ab$. Putting $(\bar{e}_4 + \delta\bar{e}_1) \wedge (\bar{e}_5 + \delta\bar{e}_2) \wedge (\bar{e}_6 + \delta\bar{e}_3) = \alpha_1 + \delta \cdot \alpha_2$, we find $\alpha_1 = \bar{e}_4 \wedge \bar{e}_5 \wedge \bar{e}_6 + b(\bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_6 + \bar{e}_1 \wedge \bar{e}_5 \wedge \bar{e}_3 + \bar{e}_4 \wedge \bar{e}_2 \wedge \bar{e}_3) + ab \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3$ and $\alpha_2 = \bar{e}_1 \wedge \bar{e}_5 \wedge \bar{e}_6 + \bar{e}_4 \wedge \bar{e}_2 \wedge \bar{e}_6 + \bar{e}_4 \wedge \bar{e}_5 \wedge \bar{e}_3 + a(\bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_6 + \bar{e}_1 \wedge \bar{e}_5 \wedge \bar{e}_3 + \bar{e}_4 \wedge \bar{e}_2 \wedge \bar{e}_3) + (a^2 + b)\bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3$. By Propositions 5.2, 6.4 and 6.12, we may without loss of generality suppose that $\pi_{\mathbb{F}'} = \langle \bar{e}_4 + \delta\bar{e}_1, \bar{e}_5 + \delta\bar{e}_2, \bar{e}_6 + \delta\bar{e}_3 \rangle$. By Proposition 6.10(1)+(4), we may without loss of generality suppose that the hyperplane $H_{\mathbb{F}'}$ has representative vector $\frac{1-a}{b}\alpha_1 + \alpha_2$. One readily calculates that $\frac{1-a}{b}\alpha_1 + \alpha_2 = \alpha_{\mathbb{F}'}$.

(2) This follows from Claim (1) and Propositions 5.2, 6.5 and 6.12. ∎

7 The classification of the trivectors of a 6-dimensional vector space

7.1 Statement of the result

Let V be a 6-dimensional vector space over a field $\mathbb{F}$. Let $B^* = (\bar{e}_1^*, \bar{e}_2^*, \ldots, \bar{e}_6^*)$ be a given ordered basis of V and let $\overline{\mathbb{F}}$ denote a fixed algebraic closure of $\mathbb{F}$. (In fact for the discussion in this section, it suffices to take for $\overline{\mathbb{F}}$ any extension field of $\mathbb{F}$ over which all quadratic polynomials of $\mathbb{F}[X]$ split.) For every quadratic extension $\mathbb{F}_1$ of $\mathbb{F}$ contained in $\overline{\mathbb{F}}$, we will now define a certain trivector $\alpha^*_{\mathbb{F}_1}$ of V. The field $\mathbb{F}_1$ can be regarded as the splitting field of some irreducible quadratic polynomial $q(X) = X^2 - aX - b \in \mathbb{F}[X]$. Since $b = -q(0) \neq 0 \neq q(1) = 1-a-b$, the values $\mu_1 := a+b-1$ and $\mu_2 := \frac{1-a-b}{b}$ are nonzero. The field $\mathbb{F}_1$ is also the splitting field of the quadratic polynomial $\mu_2 X^2 - (\mu_1\mu_2 + \mu_1 + \mu_2)X + \mu_1 \in \mathbb{F}[X]$. Now, define

$$\alpha^*_{\mathbb{F}_1} \;:=\; \mu_1 \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \mu_2 \cdot \bar{e}_4^* \wedge \bar{e}_5^* \wedge \bar{e}_6^* + (\bar{e}_1^* + \bar{e}_4^*) \wedge (\bar{e}_2^* + \bar{e}_5^*) \wedge (\bar{e}_3^* + \bar{e}_6^*).$$

The aim of this section is to use the above-developed theory to give a classification of the trivectors of V.

Proposition 7.1 (1) *If $\mathbb{F}_1$ and $\mathbb{F}_2$ are two distinct quadratic extensions of $\mathbb{F}$ contained in $\overline{\mathbb{F}}$, then $\alpha^*_{\mathbb{F}_1}$ and $\alpha^*_{\mathbb{F}_2}$ are not equivalent.*

(2) *Every nonzero trivector of V is equivalent with precisely one of the following vectors:*

- $\alpha_1^* := \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^*$;
- $\alpha_2^* := \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \bar{e}_1^* \wedge \bar{e}_4^* \wedge \bar{e}_5^*$;
- $\alpha_3^* := \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \bar{e}_4^* \wedge \bar{e}_5^* \wedge \bar{e}_6^*$;
- $\alpha_4^* := \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_4^* + \bar{e}_1^* \wedge \bar{e}_3^* \wedge \bar{e}_5^* + \bar{e}_2^* \wedge \bar{e}_3^* \wedge \bar{e}_6^*$;
- $\alpha^*_{\mathbb{F}_1}$ *for some quadratic extension $\mathbb{F}_1$ of $\mathbb{F}$ contained in $\overline{\mathbb{F}}$.*

Remarks. (1) Proposition 7.1(1) was already obtained in Proposition 6.16(2).

(2) As told earlier, the classification of the trivectors of a 6-dimensional vector space is due to Revoy [15] for arbitrary fields and a number of other authors for some special classes of fields, see [4, 6, 10, 11, 14]. The description of the trivector $\alpha^*_{\mathbb{F}_1}$ as given in Proposition 7.1 is more symmetric than the descriptions given in [6] and [15], where a distinction has been made between the case where the extension $\mathbb{F}_1/\mathbb{F}$ is separable and the case where the extension is not separable.

The classification mentioned in Proposition 7.1(2) is in fact also a classification of the trivectors, up to semi-equivalence, as the following lemma indicates.

Lemma 7.2 (1) *Let $i \in \{1,2,3,4\}$. Then every trivector semi-equivalent with α_i^* is also equivalent with α_i^*.*

(2) *Let $\mathbb{F}_1$ be a quadratic extension of $\mathbb{F}$ contained in $\overline{\mathbb{F}}$. Then every trivector semi-equivalent with $\alpha_{\mathbb{F}_1^*}$ is also equivalent with $\alpha_{\mathbb{F}_1^*}$.*

Proof. (1) It suffices to prove that α_i^* is equivalent with $\lambda \cdot \alpha_i^*$ for every $\lambda \in \mathbb{F} \setminus \{0\}$. But this is easy. If θ is the element of $GL(V)$ mapping $\bar{e}_j^*$ to $\lambda \cdot \bar{e}_j^*$ if $j \in \{1,6\}$ and $\bar{e}_j^*$ to $\bar{e}_j^*$ if $j \in \{2,3,4,5\}$, then $\bigwedge^3(\theta)(\alpha_i^*) = \lambda \cdot \alpha_i^*$.

(2) It suffices to prove that $\alpha_{\mathbb{F}_1^*}$ is equivalent with $\lambda \cdot \alpha_{\mathbb{F}_1^*}$ for every $\lambda \in \mathbb{F} \setminus \{0\}$. This is again easy. If θ is the element of $GL(V)$ mapping $\bar{e}_j^*$ to $\lambda \cdot \bar{e}_j^*$ if $j \in \{1,4\}$ and $\bar{e}_j^*$ to $\bar{e}_j^*$ if $j \in \{2,3,5,6\}$, then $\bigwedge^3(\theta)(\alpha_{\mathbb{F}_1}^*) = \lambda \cdot \alpha_{\mathbb{F}_1}^*$. ■

Corollary 7.3 (1) *If $\mathbb{F}_1$ and $\mathbb{F}_2$ are two distinct quadratic extensions of $\mathbb{F}$ contained in $\overline{\mathbb{F}}$, then $\alpha_{\mathbb{F}_1}^*$ and $\alpha_{\mathbb{F}_2}^*$ are not semi-equivalent.*

(2) *Every nonzero trivector of V is semi-equivalent with precisely one of the following vectors: α_1^*, α_2^*, α_3^*, α_4^*, $\alpha_{\mathbb{F}_1}^*$ for some quadratic extension $\mathbb{F}_1$ of $\mathbb{F}$ contained in $\overline{\mathbb{F}}$.*

7.2 Some useful properties

In this subsection, V denotes a vector space of dimension $n \geq 4$ over a field $\mathbb{F}$.

Lemma 7.4 *Let $\alpha \in \bigwedge^{n-2} V$ and let U denote the set of all $\bar{x} \in V$ for which $\alpha \wedge \bar{x} = 0$. Then $n - \dim(U)$ is even.*

Proof. Let $(\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n)$ be an ordered basis of V and let B' be the ordered basis of $\bigwedge^{n-1} V$ whose i-th component is equal to $\beta_i := (-1)^{n+i} \bar{e}_1 \wedge \cdots \wedge \bar{e}_{i-1} \wedge \widehat{\bar{e}_i} \wedge \bar{e}_{i+1} \wedge \cdots \wedge \bar{e}_n$ ($i \in \{1, \ldots, n\}$). If we put $\bar{x} = X_1 \bar{e}_1 + \cdots + X_n \bar{e}_n$ and write $\alpha \wedge \bar{x}$ as a linear combination of the components of B', then $\alpha \wedge \bar{x} = o$ implies that the coefficients of $\beta_1, \beta_2, \ldots, \beta_n$ are equal to 0. Putting the coefficient of β_i, $i \in \{1, \ldots, n\}$, equal to 0 yields an equation (E_i) in the n unknowns $X_1, \ldots, X_n$. This coefficient is equal to the coefficient of $\bar{e}_1 \wedge \bar{e}_2 \wedge \cdots \wedge \bar{e}_n$ in the expression $\alpha \wedge \bar{x} \wedge \bar{e}_i \in \bigwedge^n V$. The system of equations determined by (E_i), $i \in \{1, \ldots, n\}$, can be written in matrix form as $M_\alpha \cdot [X_1 \cdots X_n]^T = [0 \cdots 0]^T$, where the i-th row of M_α corresponds to the equation (E_i). The (i,j)-th entry of M is equal to the coefficient of $\bar{e}_1 \wedge \bar{e}_2 \wedge \cdots \wedge \bar{e}_n$ in the expression $\alpha \wedge \bar{e}_j \wedge \bar{e}_i \in \bigwedge^n V$. Since $\alpha \wedge \bar{e}_i \wedge \bar{e}_i = 0$ and $\alpha \wedge \bar{e}_i \wedge \bar{e}_j = -\alpha \wedge \bar{e}_j \wedge \bar{e}_i$ for all $i, j \in \{1, \ldots, n\}$, the matrix M is skew-symmetric. Hence, $\mathrm{rank}(M) = n - \dim(U)$ must be even. ■

Corollary 7.5 *Let $\alpha \in \bigwedge^{n-3} V$ and $\bar{x} \in V$. Let $U_{\bar{x}}$ denote the set of all $\bar{y} \in V$ for which $\alpha \wedge \bar{x} \wedge \bar{y} = 0$. Then $\dim(U_{\bar{x}}) \geq 1$ and $n - \dim(U_{\bar{x}})$ is even.*

Proof. If $\bar{x} = \bar{o}$, then $\dim(U_{\bar{x}}) = n \geq 1$. If $\bar{x} \neq \bar{o}$, then $\dim(U_{\bar{x}}) \geq 1$ since $\bar{x} \in U_{\bar{x}}$. ■

For every $i \in \{0, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$ and every $\alpha \in \bigwedge^{n-3} V$, let $X_i(\alpha)$ denote the set of all points $\langle \bar{x} \rangle$ of $\mathrm{PG}(V)$ for which the dimension of the subspace $\{\bar{y} \in V \mid \alpha \wedge \bar{x} \wedge \bar{y} = 0\}$ is equal to $n - 2i$.

Lemma 7.6 *If $\alpha_1, \alpha_2 \in \bigwedge^{n-3} V$ are semi-equivalent, then there exists a projectivity η of $\mathrm{PG}(V)$ mapping $X_i(\alpha_1)$ to $X_i(\alpha_2)$ for every $i \in \{0, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$.*

Proof. Let $\theta \in GL(V)$ and $\lambda \in \mathbb{F} \setminus \{0\}$ such that $\lambda \cdot \alpha_2 = \bigwedge^{n-3}(\theta)(\alpha_1)$. Let η denote the projectivity of $\mathrm{PG}(V)$ associated to θ and let $i \in \{0, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$. Then for a point $\langle \bar{x} \rangle$ of $\mathrm{PG}(V)$, we have $\alpha_1 \wedge \bar{x} \wedge \bar{y} = 0 \Leftrightarrow \bigwedge^{n-3}(\theta)(\alpha_1) \wedge \theta(\bar{x}) \wedge \theta(\bar{y}) = 0 \Leftrightarrow \alpha_2 \wedge \theta(\bar{x}) \wedge \theta(\bar{y}) = 0$. So, $\langle \bar{x} \rangle \in X_i(\alpha)$ if and only if $\langle \bar{x} \rangle^\eta = \langle \theta(\bar{x}) \rangle \in X_i(\alpha_2)$. Hence, $X_i(\alpha_2) = X_i(\alpha_1)^\eta$. ■

7.3 Some properties of the trivectors α_1^*, α_2^*, α_3^*, α_4^* and $\alpha_{\mathbb{F}_1}^*$

Let V be a 6-dimensional vector space over a field $\mathbb{F}$ with ordered basis $(\bar{e}_1^*, \bar{e}_2^*, \ldots, \bar{e}_6^*)$. Put $\alpha_1^* := \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^*$, $\alpha_2^* := \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \bar{e}_1^* \wedge \bar{e}_4^* \wedge \bar{e}_5^*$, $\alpha_3^* := \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \bar{e}_4^* \wedge \bar{e}_5^* \wedge \bar{e}_6^*$ and $\alpha_4^* := \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_4^* + \bar{e}_1^* \wedge \bar{e}_3^* \wedge \bar{e}_5^* + \bar{e}_2^* \wedge \bar{e}_3^* \wedge \bar{e}_6^*$. Put $\alpha_i := \alpha_i^* \wedge (\delta_1 \bar{e}_1^* + \delta_2 \bar{e}_2^* + \cdots + \delta_6 \bar{e}_6^*)$ and let $M_i := M_{\alpha_i}$ denote the matrix as defined in the proof of Lemma 7.4. We find

$$M_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\delta_6 & \delta_5 \\ 0 & 0 & 0 & \delta_6 & 0 & -\delta_4 \\ 0 & 0 & 0 & -\delta_5 & \delta_4 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\delta_6 & 0 & 0 & \delta_3 \\ 0 & \delta_6 & 0 & 0 & 0 & -\delta_2 \\ 0 & 0 & 0 & 0 & -\delta_6 & \delta_5 \\ 0 & 0 & 0 & \delta_6 & 0 & -\delta_4 \\ 0 & -\delta_3 & \delta_2 & -\delta_5 & \delta_4 & 0 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} 0 & \delta_3 & -\delta_2 & 0 & 0 & 0 \\ -\delta_3 & 0 & \delta_1 & 0 & 0 & 0 \\ \delta_2 & -\delta_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\delta_6 & \delta_5 \\ 0 & 0 & 0 & \delta_6 & 0 & -\delta_4 \\ 0 & 0 & 0 & -\delta_5 & \delta_4 & 0 \end{bmatrix}, M_4 = \begin{bmatrix} 0 & 0 & 0 & \delta_5 & -\delta_4 & 0 \\ 0 & 0 & 0 & \delta_6 & 0 & -\delta_4 \\ 0 & 0 & 0 & 0 & \delta_6 & -\delta_5 \\ -\delta_5 & -\delta_6 & 0 & 0 & \delta_1 & \delta_2 \\ \delta_4 & 0 & -\delta_6 & -\delta_1 & 0 & \delta_3 \\ 0 & \delta_4 & \delta_5 & -\delta_2 & -\delta_3 & 0 \end{bmatrix}.$$

So,

- $X_0(\alpha_1^*) = \langle \bar{e}_1^*, \bar{e}_2^*, \bar{e}_3^* \rangle$, $X_1(\alpha_1^*) = \mathrm{PG}(V) \setminus \langle \bar{e}_1^*, \bar{e}_2^*, \bar{e}_3^* \rangle$ and $X_2(\alpha_1^*) = \emptyset$;
- $X_0(\alpha_2^*) = \{\langle \bar{e}_1^* \rangle\}$, $X_1(\alpha_2^*) = \langle \bar{e}_1^*, \bar{e}_2^*, \bar{e}_3^*, \bar{e}_4^*, \bar{e}_5^* \rangle \setminus \{\langle \bar{e}_1^* \rangle\}$ and $X_2(\alpha_2^*) = \mathrm{PG}(V) \setminus \langle \bar{e}_1^*, \bar{e}_2^*, \bar{e}_3^*, \bar{e}_4^*, \bar{e}_5^* \rangle$;
- $X_0(\alpha_3^*) = \emptyset$, $X_1(\alpha_3^*) = \langle \bar{e}_1^*, \bar{e}_2^*, \bar{e}_3^* \rangle \cup \langle \bar{e}_4^*, \bar{e}_5^*, \bar{e}_6^* \rangle$ and $X_2(\alpha_3^*) = \mathrm{PG}(V) \setminus (\langle \bar{e}_1^*, \bar{e}_2^*, \bar{e}_3^* \rangle \cup \langle \bar{e}_4^*, \bar{e}_5^*, \bar{e}_6^* \rangle)$;
- $X_0(\alpha_4^*) = \emptyset$, $X_1(\alpha_4^*) = \langle \bar{e}_1^*, \bar{e}_2^*, \bar{e}_3^* \rangle$ and $X_2(\alpha_4^*) = \mathrm{PG}(V) \setminus \langle \bar{e}_1^*, \bar{e}_2^*, \bar{e}_3^* \rangle$.

With the aid of Lemma 7.6, we obtain that α_1^*, α_2^*, α_3^* and α_4^* are mutually nonequivalent. We will now also show that each of $\alpha_1^*, \alpha_2^*, \alpha_3^*, \alpha_4^*$ is nonequivalent with $\alpha_{\mathbb{F}_1}^*$ for every quadratic extension $\mathbb{F}_1$ of $\mathbb{F}$ which is contained in some fixed algebraic closure $\overline{\mathbb{F}}$ of $\mathbb{F}$. As in Section 7.1, suppose that $\mathbb{F}_1$ is the splitting field of the polynomial $\mu_2 X^2 - (\mu_1 \mu_2 +$

$\mu_1 + \mu_2)X + \mu_1 \in \mathbb{F}[X]$. Put $\alpha = \alpha^*_{\mathbb{F}_1} \wedge (\delta_1 \bar{e}^*_1 + \delta_2 \bar{e}^*_2 + \cdots + \delta_6 \bar{e}^*_6)$ and let $M = M_\alpha$ denote the matrix as defined in the proof of Lemma 7.4. Then M is equal to

$$\begin{bmatrix} 0 & (\mu_2+1)\delta_3 - \delta_6 & -(\mu_2+1)\delta_2 + \delta_5 & 0 & -\delta_3 + \delta_6 & \delta_2 - \delta_5 \\ -(\mu_2+1)\delta_3 + \delta_6 & 0 & (\mu_2+1)\delta_1 - \delta_4 & \delta_3 - \delta_6 & 0 & -\delta_1 + \delta_4 \\ (\mu_2+1)\delta_2 - \delta_5 & -(\mu_2+1)\delta_1 + \delta_4 & 0 & \delta_5 - \delta_2 & \delta_1 - \delta_4 & 0 \\ 0 & \delta_6 - \delta_3 & \delta_2 - \delta_5 & 0 & -(\mu_1+1)\delta_6 + \delta_3 & (\mu_1+1)\delta_5 - \delta_2 \\ \delta_3 - \delta_6 & 0 & \delta_4 - \delta_1 & (\mu_1+1)\delta_6 - \delta_3 & 0 & -(\mu_1+1)\delta_4 + \delta_1 \\ \delta_5 - \delta_2 & \delta_1 - \delta_4 & 0 & -(\mu_1+1)\delta_5 + \delta_2 & (\mu_1+1)\delta_4 - \delta_1 & 0 \end{bmatrix}.$$

We will prove that the rank of M is always equal to 4, except when $\delta_1 = \delta_2 = \ldots = \delta_6 = 0$ in which case M has rank 0.

Suppose the rank of M is distinct from 4 and hence equal to 0 or 2. Let $i_1, j_1 \in \{1, \ldots, 6\}$ with $|i_1 - j_1| \notin \{0, 3\}$. Suppose that the two 0's which occur in row i_1 of M occur in columns j_2 and j_3. Suppose the two 0's which occur in column j_1 of M occur in rows i_2 and i_3. Now, consider the (3×3)-submatrix of M build on the rows i_1, i_2, i_3 and the columns j_1, j_2, j_3. Making use of the irreducibility of the polynomial $\mu_2 X^2 - (\mu_1\mu_2 + \mu_1 + \mu_2)X + \mu_1 \in \mathbb{F}[X]$, one can easily show that the determinant of this submatrix is equal to 0 if and only if the (i_1, j_1)-th entry of M is equal to 0. We give two examples.

(a) Suppose $i_1 = 1$ and $j_1 = 2$. Then $\{j_2, j_3\} = \{1, 4\}$ and $\{i_2, i_3\} = \{2, 5\}$. The corresponding submatrix of M is equal to

$$\begin{bmatrix} 0 & (\mu_2+1)\delta_3 - \delta_6 & 0 \\ -(\mu_2+1)\delta_3 + \delta_6 & 0 & \delta_3 - \delta_6 \\ \delta_3 - \delta_6 & 0 & (\mu_1+1)\delta_6 - \delta_3 \end{bmatrix}.$$

The determinant of this matrix is equal to $(\delta_6 - (\mu_2+1)\delta_3)(\mu_2\delta_3^2 + \mu_1\delta_6^2 - (\mu_1\mu_2 + \mu_1 + \mu_2)\delta_3\delta_6)$ which is equal to 0 if and only if $\delta_6 - (\mu_2 + 1)\delta_3 = 0$.

(b) Suppose $i_1 = 1$ and $j_1 = 5$. Then $\{j_2, j_3\} = \{1, 4\}$ and $\{i_2, i_3\} = \{2, 5\}$. The corresponding submatrix of M is equal to

$$\begin{bmatrix} 0 & 0 & -\delta_3 + \delta_6 \\ -(\mu_2+1)\delta_3 + \delta_6 & \delta_3 - \delta_6 & 0 \\ \delta_3 - \delta_6 & (\mu_1+1)\delta_6 - \delta_3 & 0 \end{bmatrix}.$$

The determinant of this matrix is equal to $(\delta_6 - \delta_3)(\mu_2\delta_3^2 + \mu_1\delta_6^2 - (\mu_1\mu_2 + \mu_1 + \mu_2)\delta_3\delta_6)$ which is equal to 0 if and only if $\delta_6 - \delta_3 = 0$.

So, all entries of the matrix M must be equal to 0. This implies that $\delta_1 = \delta_2 = \delta_3 = \delta_4 = \delta_5 = \delta_6 = 0$.

We can now conclude that $X_0(\alpha^*_{\mathbb{F}_1}) = \emptyset$, $X_1(\alpha^*_{\mathbb{F}_1}) = \emptyset$ and $X_2(\alpha^*_{\mathbb{F}_1}) = \mathrm{PG}(V)$. Lemma 7.6 then implies that $\alpha^*_{\mathbb{F}_1}$ is nonequivalent with each of $\alpha^*_1, \alpha^*_2, \alpha^*_3, \alpha^*_4$.

Lemma 7.7 *Let B be an ordered basis of V and let α be one of the trivectors $\alpha^*_1, \alpha^*_2, \alpha^*_3, \alpha^*_4$, $\alpha^*_{\mathbb{F}_1}$, where $\mathbb{F}_1$ is some quadratic extension of $\mathbb{F}$ contained in $\overline{\mathbb{F}}$. Then the dual vector of α with respect to B is equivalent to α.*

Proof. In view of Propositions 3.7 and 7.2, we may suppose that $B = B^*$. The dual vector of α_1^* with respect to B^* is equal to $\bar{e}_4^* \wedge \bar{e}_5^* \wedge \bar{e}_6^*$ which is (semi-)equivalent with α_1^*. The dual vector of α_2^* with respect to B^* is equal to $\alpha_4^* \wedge \bar{e}_5^* \wedge \bar{e}_6^* + \bar{e}_2^* \wedge \bar{e}_3^* \wedge \bar{e}_6^*$ which is (semi-)equivalent with α_2^*. The dual vector of α_3^* with respect to B^* is equal to $\bar{e}_4^* \wedge \bar{e}_5^* \wedge \bar{e}_6^* - \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^*$ which is (semi-)equivalent with α_3^*. The dual vector of α_4^* with respect to B^* is equal to $-\bar{e}_3^* \wedge \bar{e}_5^* \wedge \bar{e}_6^* - \bar{e}_2^* \wedge \bar{e}_4^* \wedge \bar{e}_6^* - \bar{e}_1^* \wedge \bar{e}_4^* \wedge \bar{e}_5^*$ which is (semi-)equivalent with α_4^*. Finally, the dual vector of $\alpha_{\mathbb{F}_1}^*$ with respect to B^* is equal to $\mu_1 \cdot \bar{e}_4^* \wedge \bar{e}_5^* \wedge \bar{e}_6^* - \mu_2 \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + (-\bar{e}_1^* + \bar{e}_4^*) \wedge (-\bar{e}_2^* + \bar{e}_5^*) \wedge (-\bar{e}_3^* + \bar{e}_6^*) = \mu_1 \cdot \bar{e}_4^* \wedge \bar{e}_5^* \wedge \bar{e}_6^* + \mu_2 \cdot (-\bar{e}_1^*) \wedge (-\bar{e}_2^*) \wedge (-\bar{e}_3^*) + (-\bar{e}_1^* + \bar{e}_4^*) \wedge (-\bar{e}_2^* + \bar{e}_5^*) \wedge (-\bar{e}_3^* + \bar{e}_6^*)$ which is (semi)-equivalent with $\alpha_{\mathbb{F}_1}^*$. ■

7.4 The classification of the trivectors

Let V be a 6-dimensional vector space over a field $\mathbb{F}$. Suppose α is a trivector of V. Let H denote the hyperplane of $A_{5,3}(\mathbb{F})$ for which α is a representative vector. We can distinguish 3 cases: (1) $X_0(\alpha) \neq \emptyset$; (2) $X_0(\alpha) = \emptyset$ and $X_1(\alpha) \neq \emptyset$; (3) $X_0(\alpha) = X_1(\alpha) = \emptyset$. If case (3) occurs, then α is called a *special trivector*.

(I) Suppose $X_0(\alpha) = \emptyset$. Then there exists a nonzero vector $\bar{e}_1 \in V$ such that $\alpha \wedge \bar{e}_1 = 0$. Then $\alpha = \bar{e}_1 \wedge \beta$ for some $\beta \in \bigwedge^2 V$. By Proposition 4.1(1), there exist linearly independent vectors $\bar{e}_2, \bar{e}_3, \bar{e}_4, \bar{e}_5 \in V$ such that α is equal to either $\bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3$ or $\bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 + \bar{e}_1 \wedge \bar{e}_4 \wedge \bar{e}_5$. In the former case, α is equivalent with α_1^*. In the latter case, α is equivalent with either α_1^* or α_2^* depending on whether $\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_5$ are linearly dependent or not.

(II) Suppose $X_0(\alpha) = \emptyset$ and $X_1(\alpha) \neq \emptyset$. Let $\langle \bar{e}_1 \rangle \in X_1(\alpha)$ such that $\{\bar{x} \in V \mid \alpha \wedge \bar{e}_1 \wedge \bar{x} = 0\}$ has dimension 4. Then $\alpha \wedge \bar{e}_1 = \bar{x} \wedge \bar{y} \wedge \bar{z} \wedge \bar{e}_1$ for some linearly independent vectors $\bar{x}, \bar{y}, \bar{z}$ of V satisfying $\bar{e}_1 \notin \langle \bar{x}, \bar{y}, \bar{z} \rangle$. Since $(\alpha - \bar{x} \wedge \bar{y} \wedge \bar{z}) \wedge \bar{e}_1 = 0$, there exists by (I) a 4-dimensional subspace $\langle \bar{e}_2, \bar{e}_3, \bar{e}_4, \bar{e}_5 \rangle$ of V not containing $\bar{e}_1$ such that α is equal to either $\bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 + \bar{x} \wedge \bar{y} \wedge \bar{z}$ or $\bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 + \bar{e}_1 \wedge \bar{e}_4 \wedge \bar{e}_5 + \bar{x} \wedge \bar{y} \wedge \bar{z}$. In the former case, the fact that $X_0(\alpha) = \emptyset$ implies that the 3-spaces $\langle \bar{e}_1, \bar{e}_2, \bar{e}_3 \rangle$ and $\langle \bar{x}, \bar{y}, \bar{z} \rangle$ of V are disjoint. So, in this case α is equivalent with α_3^*. Suppose $\alpha = \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 + \bar{e}_1 \wedge \bar{e}_4 \wedge \bar{e}_5 + \bar{x} \wedge \bar{y} \wedge \bar{z}$. By Section 4.2 and the fact that $X_0(\alpha) = \emptyset$, the 3-dimensional subspace $\langle \bar{x}, \bar{y}, \bar{z} \rangle$ is not contained in $\langle \bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4, \bar{e}_5 \rangle$. So, $\langle \bar{x}, \bar{y}, \bar{z} \rangle \cap \langle \bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4, \bar{e}_5 \rangle = \langle \bar{u}, \bar{v} \rangle$ for some linearly independent vectors $\bar{u}$ and $\bar{v}$ of $\langle \bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4, \bar{e}_5 \rangle$ satisfying $\bar{e}_1 \notin \langle \bar{u}, \bar{v} \rangle$ (otherwise $\langle \bar{e}_1 \rangle \in X_0(\alpha)$). Since $\bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 + \bar{e}_1 \wedge \bar{e}_4 \wedge \bar{e}_5 = \bar{e}_1 \wedge (\bar{e}_2 + \lambda_2 \bar{e}_1) \wedge (\bar{e}_3 + \lambda_3 \bar{e}_1) + \bar{e}_1 \wedge (\bar{e}_4 + \lambda_4 \bar{e}_1) \wedge (\bar{e}_5 + \lambda_5 \bar{e}_1)$ for all $\lambda_2, \lambda_3, \lambda_4, \lambda_5 \in \mathbb{F}$, we may without loss of generality suppose that $\bar{u}, \bar{v} \in \langle \bar{e}_2, \bar{e}_3, \bar{e}_4, \bar{e}_5 \rangle$. By Proposition 4.1(2), α is equal to $\bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 + \bar{e}_1 \wedge \bar{e}_4 \wedge \bar{e}_5 + \bar{e}_2 \wedge \bar{e}_3 \wedge \bar{e}_6$ or $\bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 + \bar{e}_1 \wedge \bar{e}_4 \wedge \bar{e}_5 + \bar{e}_2 \wedge \bar{e}_4 \wedge \bar{e}_6$ for some $\bar{e}_6 \in V \setminus \langle \bar{e}_1, \bar{e}_2, \ldots, \bar{e}_5 \rangle$ satisfying $\langle \bar{x}, \bar{y}, \bar{z} \rangle = \langle \bar{e}_2, \bar{e}_3, \bar{e}_6 \rangle$ (former case) or $\langle \bar{x}, \bar{y}, \bar{z} \rangle = \langle \bar{e}_2, \bar{e}_4, \bar{e}_6 \rangle$ (latter case). In the former case, $\alpha = \bar{e}_2 \wedge \bar{e}_3 \wedge (\bar{e}_1 + \bar{e}_6) + \bar{e}_1 \wedge \bar{e}_4 \wedge \bar{e}_5$ is equivalent with α_3^*. In the latter case, α is equivalent with α_4^*.

(III) Suppose α is a special trivector of U. Let S denote the set of all lines $\langle \bar{v}_1, \bar{v}_2 \rangle$ of $\mathrm{PG}(V)$ for which $\alpha \wedge \bar{v}_1 \wedge \bar{v}_2 = 0$. For every point $p = \langle \bar{x} \rangle$ of $\mathrm{PG}(V)$, $\{\bar{y} \in V \mid \alpha \wedge \bar{x} \wedge \bar{y} = 0\}$

is a two-dimensional subspace of V containing $\bar{x}$ since $p \in X_2(\alpha)$. Hence, S is a spread of $\mathrm{PG}(V)$. Moreover, every plane through a line of S belongs to the hyperplane H. Now, let B be a given ordered basis of V and let α' denote the dual vector of α with respect to B. By (I)+(II), Proposition 3.8 and Lemma 7.7, α' is a special trivector of V. So, if S' denotes the set of all lines $\langle \bar{v}_1, \bar{v}_2 \rangle$ of $\mathrm{PG}(V)$ for which $\alpha' \wedge \bar{v}_1 \wedge \bar{v}_2 = 0$ and if H' denotes the hyperplane of $A_{5,3}(\mathbb{F})$ for which α' is a representative vector, then also S' is a spread of $\mathrm{PG}(V)$ and every plane through a line of S' belongs to H'. Hence, by Corollary 3.6, there exists a set R of 3-dimensional subspaces of $\mathrm{PG}(V)$ satisfying the following properties: (1) every 4-dimensional subspace of $\mathrm{PG}(V)$ contains a unique element of R; (2) all planes contained in an element of R belong to H.

We prove that it is impossible that there is some line $L \in S$ and some 3-dimensional subspace $\pi \in R$ which intersect in a unique point p. Recall that every plane through L belongs to H and that every plane of π through p belongs to H. Since H is a hyperplane of $A_{n-1,k}(\mathbb{F})$, it readily follows that every plane of $\langle \pi, L \rangle$ through p belongs to H. Now, let K_1 be an arbitrary line through p not contained in $\langle \pi, L \rangle$. Since H is a hyperplane of $A_{5,3}(\mathbb{F})$, there are two distinct planes of H through K_1. These two planes intersect $\langle \pi, L \rangle$ in two distinct lines, say K_2 and K_3. Recall that every plane of $\langle x, \pi \rangle$ through K_i, $i \in \{2,3\}$, belongs to H. Since also $\langle K_1, K_i \rangle$, $i \in \{1,2\}$, belongs to the hyperplane H, every plane through K_i, $i \in \{1,2\}$, belongs to H. This implies that K_1 and K_2 belong to S, a contradiction, since only 1 line through p belongs to S.

We prove that the spread S satisfies property (R1) of Section 6.1. Let L_1 and L_2 be two distinct lines of S and let π' be an arbitrary 4-dimensional subspace of $\mathrm{PG}(V)$ containing L_1 and L_2. Then π' contains a unique element π of R. The lines L_1 and L_2 meet π and hence are contained in π by the previous paragraph. So, $\pi = \langle L_1, L_2 \rangle$. If $p \in \pi$, then the unique line of S through p is contained in π by the previous paragraph. So, the lines of S contained in π determine a spread of π.

We prove that the spread S satisfies property (R2) of Section 6.1. Let L_1, L_2 and L_3 be three distinct lines which are contained in some 3-dimensional subspace of $\mathrm{PG}(V)$. We can choose vectors $\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4$ of V such that $L_1 = \langle \bar{e}_1, \bar{e}_2 \rangle$, $L_2 = \langle \bar{e}_3, \bar{e}_4 \rangle$ and $L_3 = \langle \bar{e}_1 + \bar{e}_3, \bar{e}_2 + \bar{e}_4 \rangle$. Then $M_1 = \langle \bar{e}_1, \bar{e}_3 \rangle$, $M_2 = \langle \bar{e}_2, \bar{e}_4 \rangle$ and $M_3 = \langle \bar{e}_1 + \bar{e}_2, \bar{e}_3 + \bar{e}_4 \rangle$ are lines meeting L_1, L_2 and L_3. So, $\mathcal{R}(L_1, L_2, L_3)$ consists of those lines of $\mathrm{PG}(V)$ which meet M_1, M_2 and M_3. These are the lines $L_1 = \langle \bar{e}_1, \bar{e}_2 \rangle$ and $K_\lambda = \langle \lambda \bar{e}_1 + \bar{e}_3, \lambda \bar{e}_2 + \bar{e}_4 \rangle$. Now, the facts that L_1, L_2 and L_3 belong to S imply that $\alpha \wedge \bar{e}_1 \wedge \bar{e}_2 = \alpha \wedge \bar{e}_3 \wedge \bar{e}_4 = \alpha \wedge (\bar{e}_1 + \bar{e}_3) \wedge (\bar{e}_2 + \bar{e}_4) = 0$, or equivalently, $\alpha \wedge \bar{e}_1 \wedge \bar{e}_2 = \alpha \wedge \bar{e}_3 \wedge \bar{e}_4 = \alpha \wedge (\bar{e}_1 \wedge \bar{e}_4 + \bar{e}_3 \wedge \bar{e}_2) = 0$. Now, $K_\lambda \in S$ since $\alpha \wedge (\lambda \bar{e}_1 + \bar{e}_3) \wedge (\lambda \bar{e}_2 + \bar{e}_4) = \lambda^2 (\alpha \wedge \bar{e}_1 \wedge \bar{e}_2) + \lambda \cdot \alpha \wedge (\bar{e}_1 \wedge \bar{e}_4 + \bar{e}_3 \wedge \bar{e}_2) + \alpha \wedge \bar{e}_3 \wedge \bar{e}_4 = 0$.

We can conclude that S is a regular spread of $\mathrm{PG}(V)$. Since H is a hyperplane of $A_{5,3}(\mathbb{F})$ containing all planes which contain a line of S, the representative vector α of H must be equivalent to $\alpha^*_{\mathbb{F}_1}$ for some quadratic extension $\mathbb{F}_1$ of $\mathbb{F}$ contained in $\overline{\mathbb{F}}$.

7.5 Applications to hyperplanes of $A_{5,3}(\mathbb{F})$

Proposition 7.8 (1) *For every hyperplane H of $A_{5,3}(\mathbb{F})$, there is an automorphism of $A_{5,3}(\mathbb{F})$ induced by a duality of* $\mathrm{PG}(V)$ *mapping H to itself.*

(2) *Let* H_1 *and* H_2 *be two hyperplanes of* $A_{5,3}(\mathbb{F})$. *Then* H_1 *and* H_2 *are isomorphic if and only if there is an automorphism of* $A_{5,3}(\mathbb{F})$ *induced by a collineation of* $\mathrm{PG}(V)$ *mapping* H_1 *to* H_2.

Proof. Claim (1) follows from Propositions 5.2, 5.5 and Lemma 7.7. Claim (2) follows from Claim (1) and Proposition 5.1(2). ∎

For every $i \in \{1, 2, 3, 4\}$, let H_i^* denote the hyperplane of $A_{5,3}(\mathbb{F})$ having α_i^* as representative vector. For every quadratic extension $\mathbb{F}_1$ of $\mathbb{F}$ which is contained in $\overline{\mathbb{F}}$, let $H_{\mathbb{F}_1}^*$ denote the hyperplane of $A_{5,3}(\mathbb{F})$ with representative vector $\alpha_{\mathbb{F}_1}^*$.

Proposition 7.9 (1) *The hyperplanes* H_1^*, H_2^*, H_3^* *and* H_4^* *are mutually nonisomorphic.*

(2) *For every quadratic extension* $\mathbb{F}_1$ *of* $\mathbb{F}$ *which is contained in* $\overline{\mathbb{F}}$, $H_{\mathbb{F}_1}^*$ *is not isomorphic to* H_1^*, H_2^*, H_3^*, *nor to* H_4^*.

(3) *If* $\mathbb{F}_1$ *and* $\mathbb{F}_2$ *are two quadratic extensions of* $\mathbb{F}_1$ *which are contained in* $\overline{\mathbb{F}}$, *then* $H_{\mathbb{F}_1}^*$ *and* $H_{\mathbb{F}_2}^*$ *are isomorphic if and only if there exist* $a, b \in \mathbb{F}$ *and an automorphism* ψ *of* $\mathbb{F}$ *such that* $\mathbb{F}_1$ *and* $\mathbb{F}_2$ *are the splitting fields of the respective polynomials* $X^2 + aX + b$ *and* $X^2 + a^\psi X + b^\psi$ *of* $\mathbb{F}[X]$.

Proof. If ψ is an automorphism of $\mathbb{F}$, then $(\alpha_i^*)^{\psi_{B^*}} = \alpha_i^*$ for every $i \in \{1, 2, 3, 4\}$. Claims (1) and (2) of the proposition then follow from Propositions 5.2, 5.4, 7.8(2) and Corollary 7.3. Claim (3) was already proved in Corollary 6.15(3). ∎

For every point p of $\mathrm{PG}(V)$, let X_p denote the set of all planes of $\mathrm{PG}(V)$ containing p. The subgeometry $\widetilde{X_p}$ of $A_{5,3}(\mathbb{F})$ induced on X_p is isomorphic to $A_{4,2}(\mathbb{F})$. We call $\widetilde{X_p}$ an $A_{4,2}(\mathbb{F})$-*subgeometry of Type I*. For every hyperplane π of $\mathrm{PG}(V)$, let Y_π denote the set of all planes of $\mathrm{PG}(V)$ contained in π. The subgeometry $\widetilde{Y_\pi}$ of $A_{5,3}(\mathbb{F})$ induced on Y_π is isomorphic to $A_{4,2}(\mathbb{F})$. We call $\widetilde{Y_\pi}$ an $A_{4,2}(\mathbb{F})$-*geometry of Type II*.

There are two isomorphism classes of hyperplanes of $A_{4,2}(\mathbb{F})$ respectively corresponding to the two equivalence classes of nonzero symplectic forms on a vector space W of dimension 5 over a field $\mathbb{F}$.

(a) Every hyperplane corresponding to a symplectic form on W whose radical is 3-dimensional consists of the lines of $\mathrm{PG}(W)$ which meet a given plane of $\mathrm{PG}(W)$. We call such a hyperplane *singular*.

(b) Every other hyperplane of $A_{4,2}(\mathbb{F})$ corresponds to a symplectic form on W whose radical is 1-dimensional.

In Section 7.3, we calculated $X_0(\alpha)$, $X_1(\alpha)$ and $X_2(\alpha)$ for the trivectors α belonging to the distinct (semi)-equivalence classes. This information can be turned into geometrical information for the corresponding hyperplanes as the following proposition indicates. This information allows us to distinguish hyperplanes by means of some of their geometrical properties.

Proposition 7.10 *Let* H *be a hyperplane of* $A_{5,3}(\mathbb{F})$ *with representative vector* α *and let* $p = \langle \bar{x} \rangle$ *be a point of* $\mathrm{PG}(V)$. *Then:*

(1) *$X_p \subseteq H$ if and only if $p \in X_0(\alpha)$;*
(2) *$X_p \cap H$ is a singular hyperplane of $\widetilde{X_p} \cong A_{4,2}(\mathbb{F})$ if and only if $p \in X_1(\alpha)$;*
(3) *$X_p \cap H$ is a nonsingular hyperplane of $\widetilde{X_p} \cong A_{4,2}(\mathbb{F})$ if and only if $p \in X_2(\alpha)$.*

Proof. Let f be the symplectic form $f_{\alpha\wedge\bar{x},\xi}$, where ξ is some nonzero vector of $\bigwedge^6 V$. The radical of the form f consists of all $\bar{y} \in V$ for which $\alpha \wedge \bar{x} \wedge \bar{y} = 0$. So, $f = 0$ if and only if $\alpha \wedge \bar{x} = 0$, or equivalently, $p \in X_0(\alpha)$. This precisely happens when $X_p \subseteq H$. Clearly, $Rad(f)$ has dimension 4 if and only if $p \in X_1(\alpha)$ and dimension 2 if and only if $p \in X_2(\alpha)$. The claims of the proposition follow. ■

As an application of Proposition 7.10, we will calculate the total number of points in a hyperplane of $A_{5,3}(\mathbb{F})$ if $\mathbb{F}$ is finite.

Proposition 7.11 *Suppose $\mathbb{F}$ be the finite field with q elements, and let H be a hyperplane of $A_{5,3}(\mathbb{F})$ with representative vector α. Then H contains $\frac{1}{q^2+q+1} \cdot \Big(|X_0(\alpha)| \cdot (q^2+1)(q^4+q^3+q^2+q+1) + |X_1(\alpha)| \cdot (q^2+q+1)(q^3+q^2+1) + |X_2(\alpha)| \cdot (q+1)(q^2+1)^2\Big)$ points.*

Proof. This follows from Proposition 7.10 and the following facts: (i) every point of $A_{5,3}(\mathbb{F})$ is contained in q^2+q+1 $A_{4,2}(\mathbb{F})$-subspaces of type I; (ii) $A_{4,2}(\mathbb{F})$ contains $(q^2+1)(q^4+q^3+q^2+q+1)$ points; (iii) a singular hyperplane of $A_{4,2}(\mathbb{F})$ contains $(q^2+q+1)(q^3+q^2+1)$ points; (iv) a nonsingular hyperplane of $A_{4,2}(\mathbb{F})$ contains $(q+1)(q^2+1)^2$ points. ■

Corollary 7.12 *Suppose $\mathbb{F}$ is a finite field with q elements and let $\mathbb{K}$ denote the unique quadratic extension of $\mathbb{F}$ contained in $\overline{\mathbb{F}}$. Then $|H_1^*| = q^8+2q^7+3q^6+3q^5+3q^4+3q^3+2q^2+q+1$, $|H_2^*| = q^8+q^7+3q^6+3q^5+3q^4+3q^3+2q^2+q+1$, $|H_3^*| = q^8+q^7+2q^6+3q^5+4q^4+3q^3+2q^2+q+1$, $|H_4^*| = q^8+q^7+2q^6+3q^5+3q^4+3q^3+2q^2+q+1$ and $|\mathbb{H}_{\mathbb{K}}^*| = q^8+q^7+2q^6+3q^5+2q^4+3q^3+2q^2+q+1$.*

Proof. This follows form Proposition 7.11 and the calculations made in Section 7.3. ■

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